Correlations of mixed systems in confining backgrounds

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Recent developments
in the holographic principle workshop

Nov 1st, 2021, Mokpo

Measures for mixed systems

Mutual information (MI)

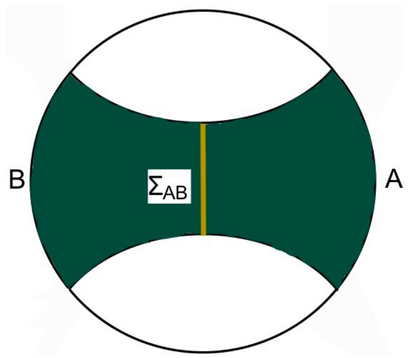
$$I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}),$$

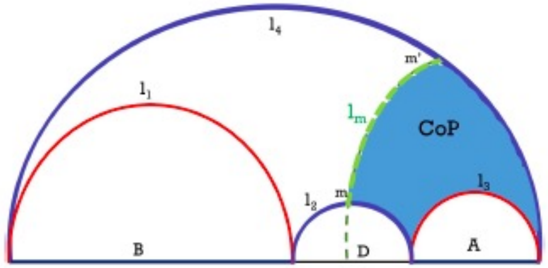
Entanglement of Purification (EoP)

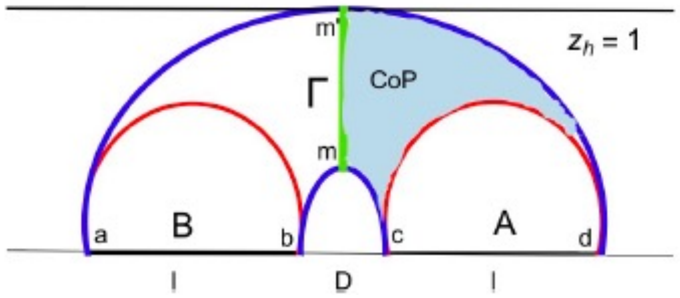
$$E_P(A:B) = \min_{\rho_{AB}=Tr_{A'B'}|\psi\rangle\langle\psi|} S(\rho_{AA'})$$

$|\psi\rangle$ is in $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}'_A \otimes \mathcal{H}'_B$

Minimal wedge cross section (EW)



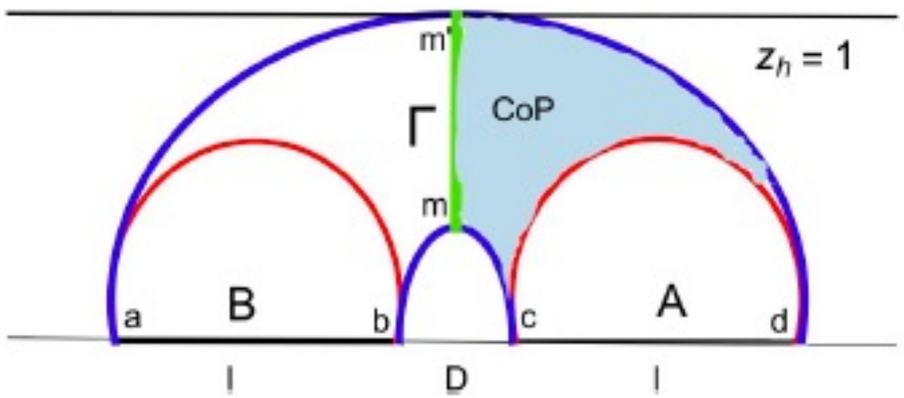




EW=EoP in various geometries

Setup: two symmetric parallel strips with width “l” and distance “D” between them.

$$A := \{l + D/2 > x_1 > D/2, -\infty < x_i < \infty, i = 2, 3, ..., d-1\}$$
$$B := \{-l - D/2 < x_1 < -D/2, -\infty < x_i < \infty, i = 2, 3, ..., d-1\}.$$



$$S_A = S_B = S(l)$$

$$S_{AB} = S(2l + D) + S(D)$$

$$I(D, l) = S_A + S_B - S_{AB} = 2S(l) - S(D) - S(2l + D)$$

EW in BTZ Black Hole Background

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$$\boxed{ds^2 = \frac{1}{z^2}\left[-f(z)dt^2 + \frac{dz^2}{f(z)} + d\vec{x}^2_{d-1}\right], \qquad f(z) := 1 - z^d/z_h^d,}$$

EW

$$\boxed{E_W = \frac{c}{3}\min\left[A^{(1)}, A^{(2)}\right],}$$

$$A^{(1)} = \log\frac{\beta}{\pi\epsilon}, \qquad A^{(2)} = \log\frac{\beta\sinh\left(\frac{\pi l}{\beta}\right)}{\pi\epsilon}$$

→ $$\sinh\left(\frac{l}{2}\right)^2 = \sinh\left(\frac{D_c}{2}\right)\sinh\left(\frac{2l+D_c}{2}\right)$$

→ $$\cosh\frac{D_c(2,l)}{2} = \sqrt{1+2\sqrt{2\cosh l}\cosh\frac{l}{2}+2\cosh l\left[\cosh\frac{3l}{2}-\sqrt{2}(\cosh l)^{3/2}\right]}$$

EW in BTZ background

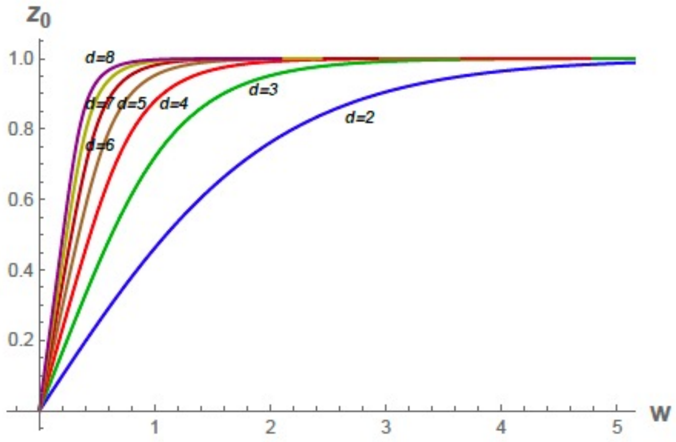
$$\boxed{\Gamma = \int_{z_D}^{z_{2l+D}} \frac{dz}{z^{d-1}\sqrt{1-\frac{z^d}{z_h^d}}},}$$

$$\frac{4}{V_{d-2}}E(l,D) = \begin{cases} \ln\frac{\tanh(\frac{D+2l}{2})}{\tanh(\frac{D}{2})} + \ln\frac{1+\sqrt{1-\frac{\tanh^2\left(\frac{D}{2}\right)}{z_h^2}}}{1+\sqrt{1-\frac{\tanh^2\left(\frac{D+2l}{2}\right)}{z_h^2}}}, & d=2, \\ \frac{-4z^{-(d-2)}\sqrt{1-\frac{z^d}{z_h^d}}+(d-4)\frac{z^2}{z_h^d}\,{}_2F_1\left(\frac{1}{2},\frac{2}{d},\frac{d+2}{d},\frac{z^d}{z_h^d}\right)}{4(d-2)}\Bigg|_{z_D}^{z_{2l+D}}, & d>2. \end{cases}$$

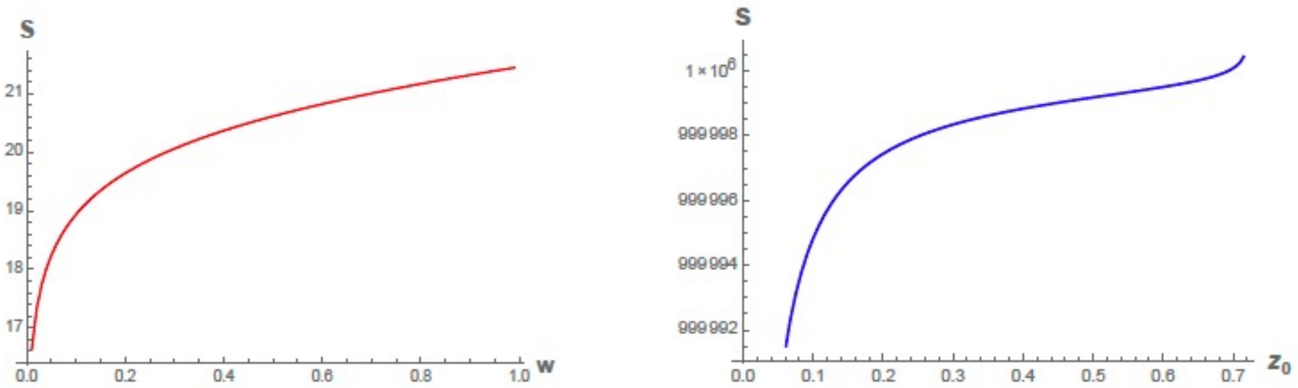
Entropy and width of one strip in the background of BTZ black brane

$$w = 2\int_{\delta}^{z_0} dz \frac{1}{\sqrt{\left(1-\frac{z^d}{z_d^d}\right)\left(\frac{z_0^{2d-2}}{z^{2d-2}}-1\right)}},$$

$$S(w) = \frac{2V_{d-2}}{4G_N}\int_{\delta}^{z_0} \frac{dz}{z^{d-1}} \frac{1}{\sqrt{\left(1-\frac{z^d}{z_h^d}\right)\left(1-\frac{z^{2d-2}}{z_0^{2d-2}}\right)}}.$$

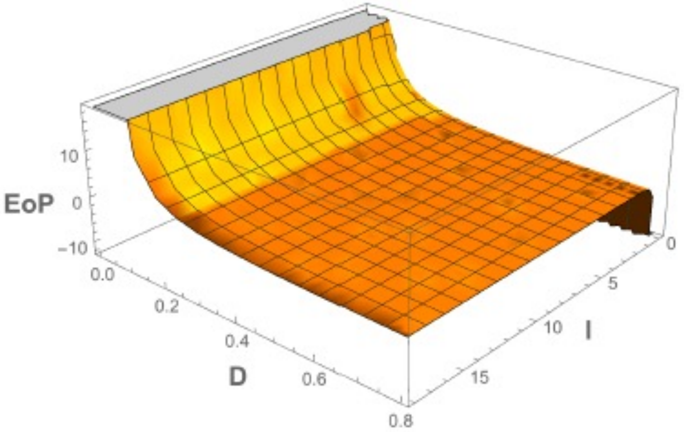


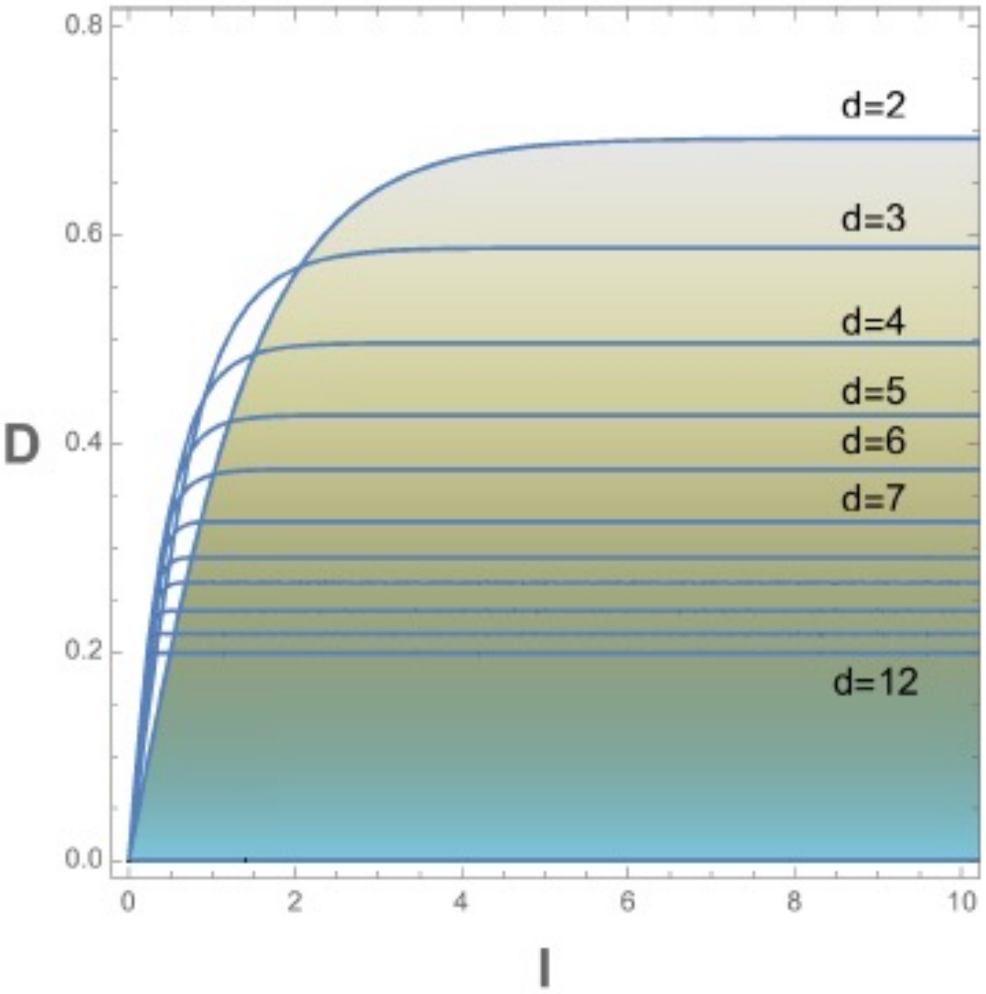
Turning point z_0 versus width w



Entanglement entropy versus “w” and “z_0”

Behavior of D_c versus "l" in BTZ background (conformal case)

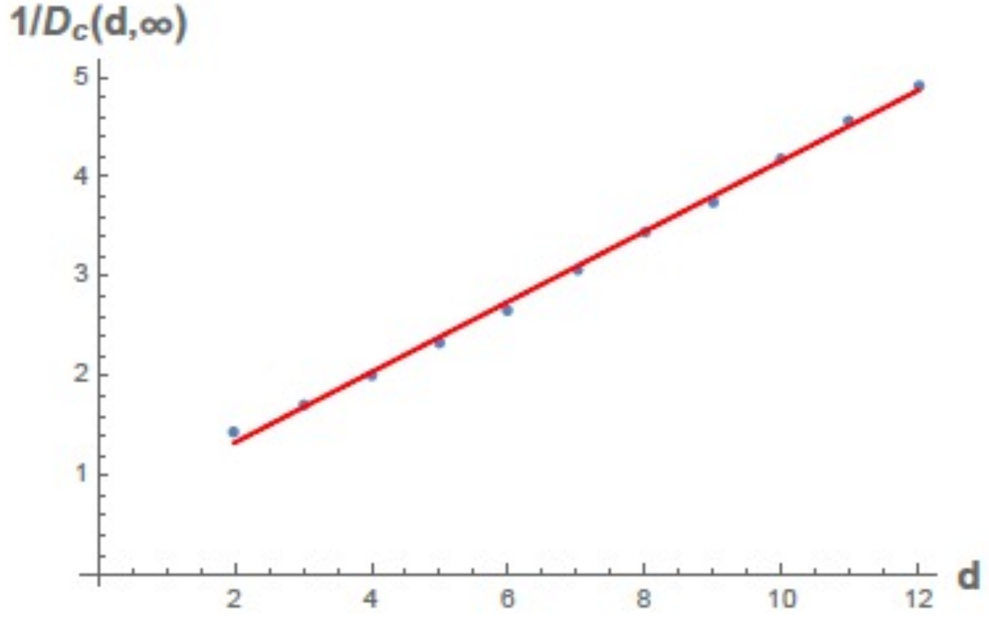


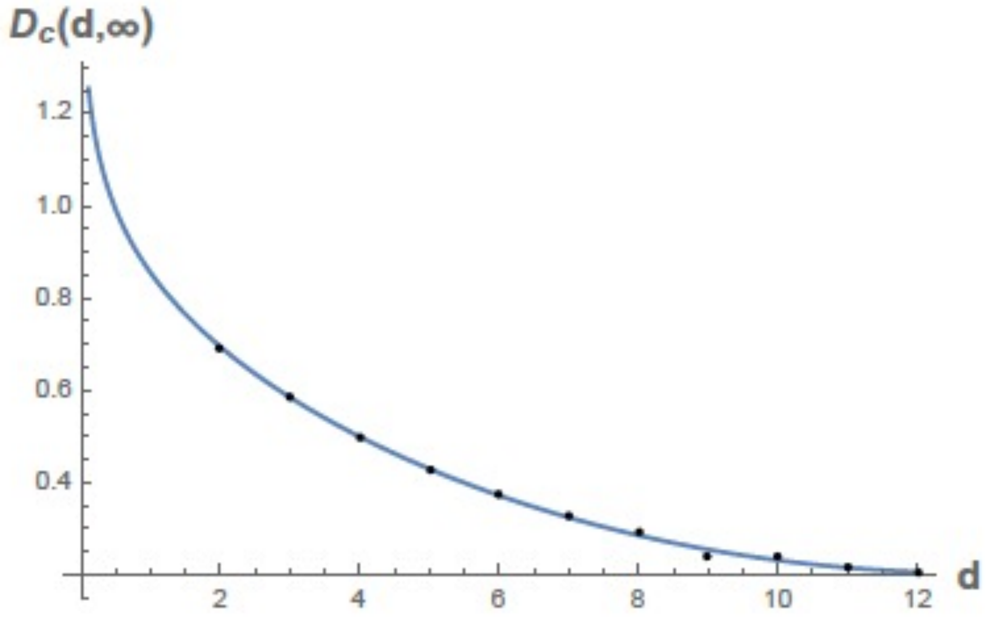


Can D_c detect phase structures?

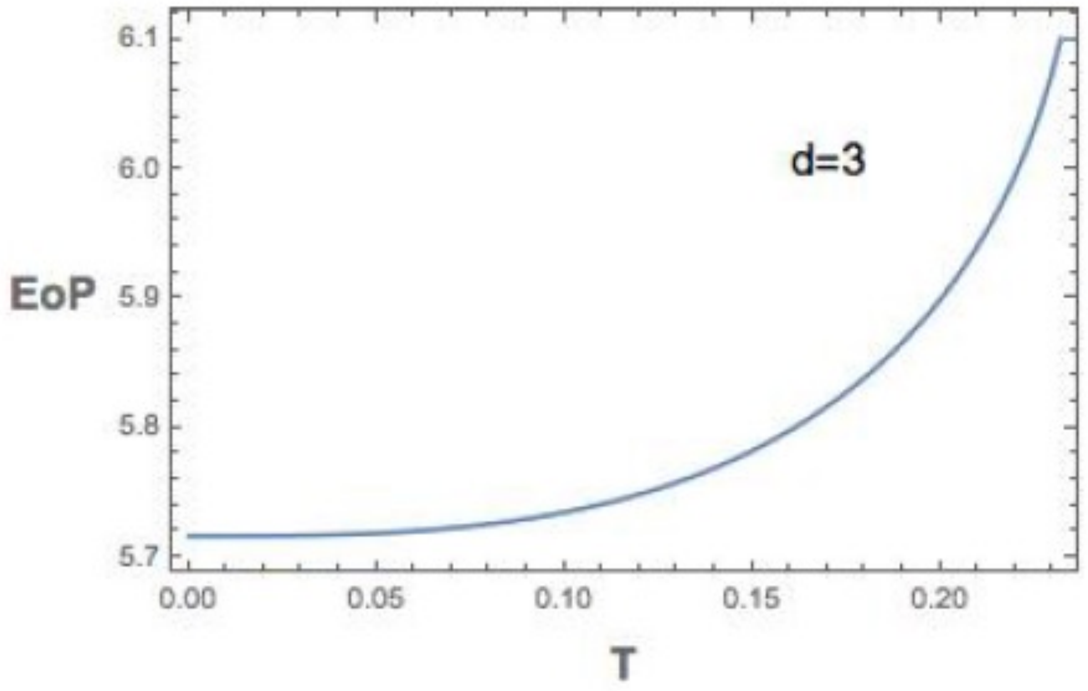
1902.02475

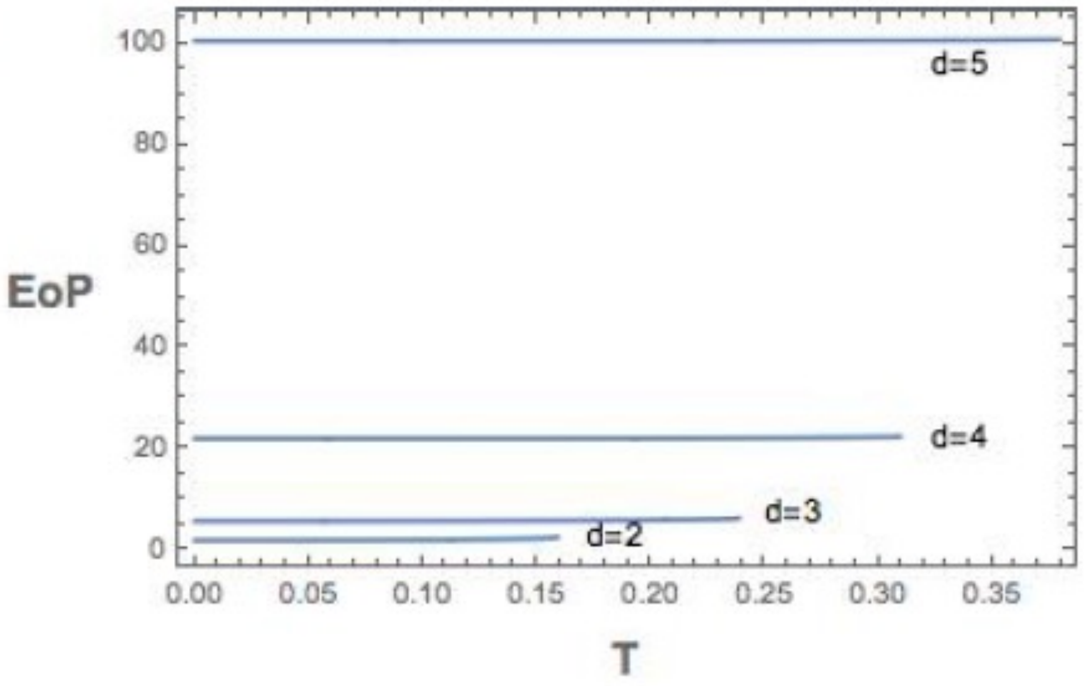
Behavior of D_c versus "d" in BTZ background (conformal case)





The behavior of EoP vs T in the background of BTZ black hole





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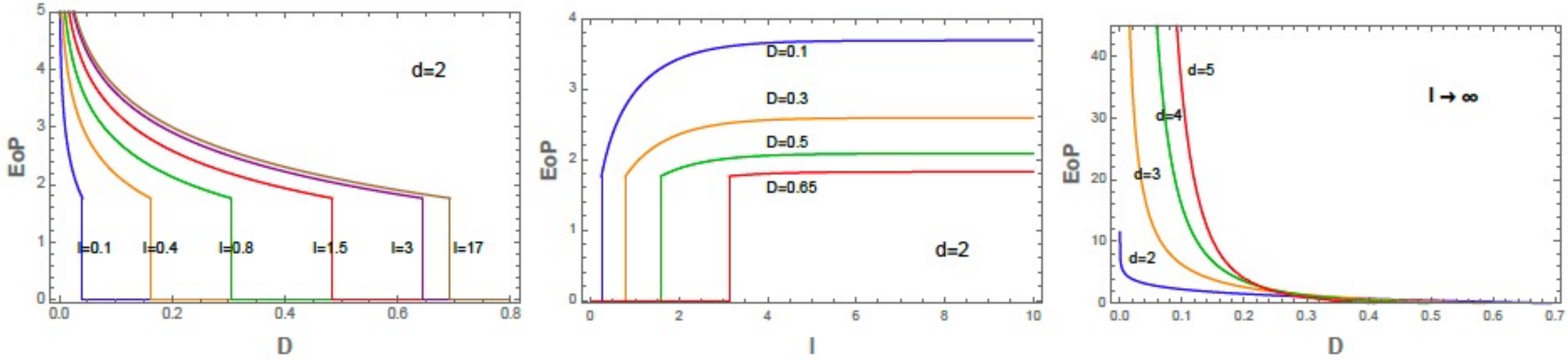
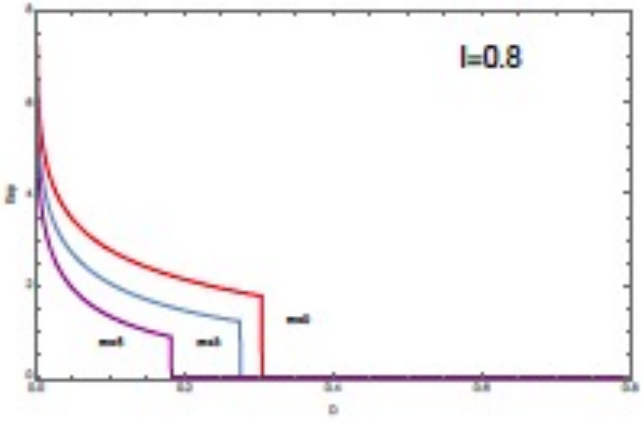
Can EoP detect phase structures?

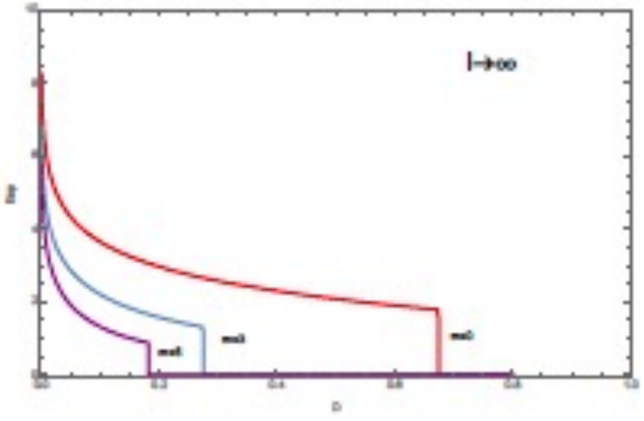
The behavior of EoP vs D and "l"

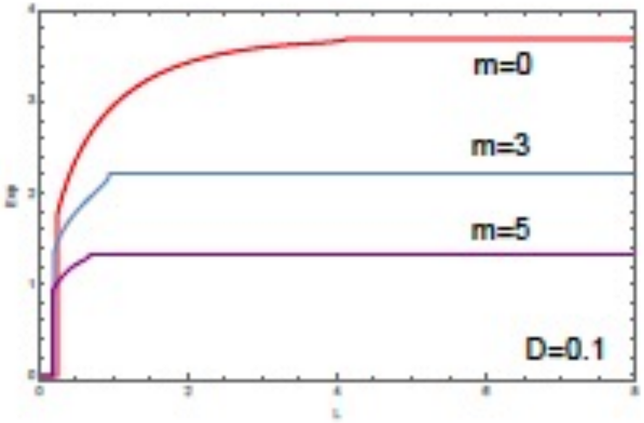


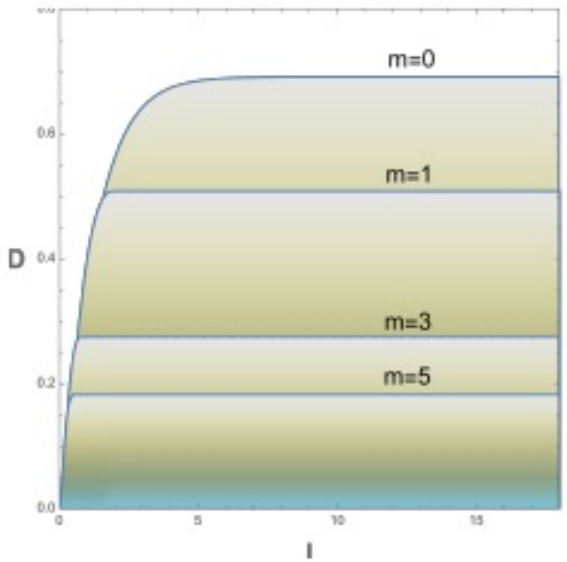
Rich information could be gathered from EoP and D_c . Let's examine it in confining backgrounds now!!

EoP and D in massive BTZ case

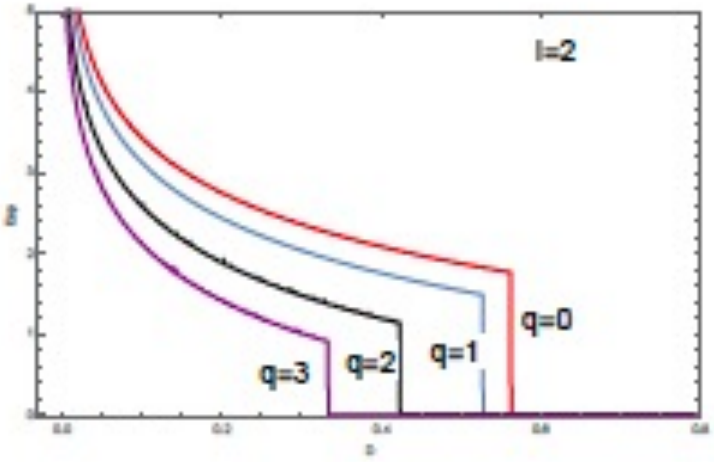


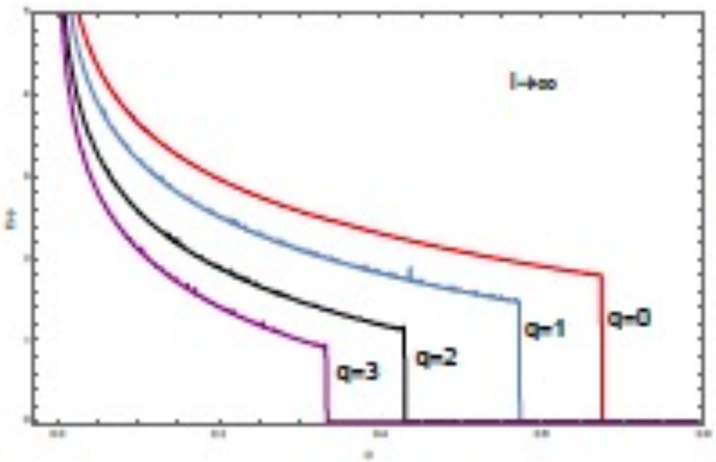


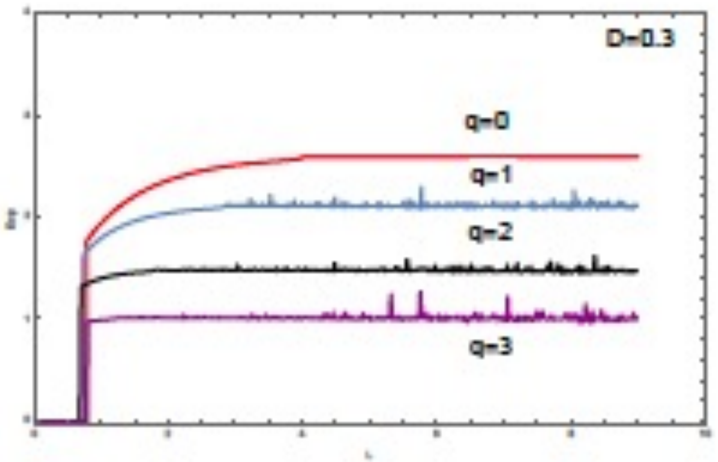


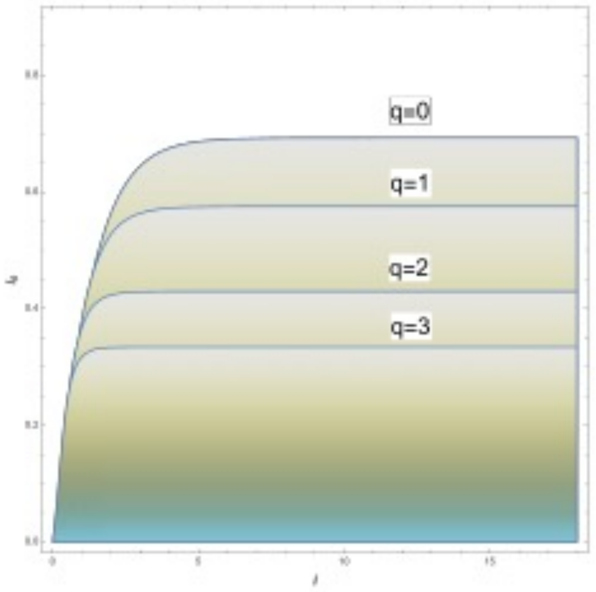


EoP and D in charged BTZ case









Confining Backgrounds

Now lets check what can we learn about the phase structrues of QCD and confining models from the “mixed correlation measures” such as MI, EoP and “D_c”.

First, the generalization of entanglement entropy (S) and EoP (EW) from conformal to confining case would be

From this →

$$S_A = \frac{1}{4G_N^{(d+2)}} \int_\gamma d^d\sigma \sqrt{G_{\text{ind}}^{(d)}},$$

To this →

$$S_A = \frac{1}{4G_N^{(10)}} \int d^8\sigma e^{-2\phi} \sqrt{G_{\text{ind}}^{(8)}},$$

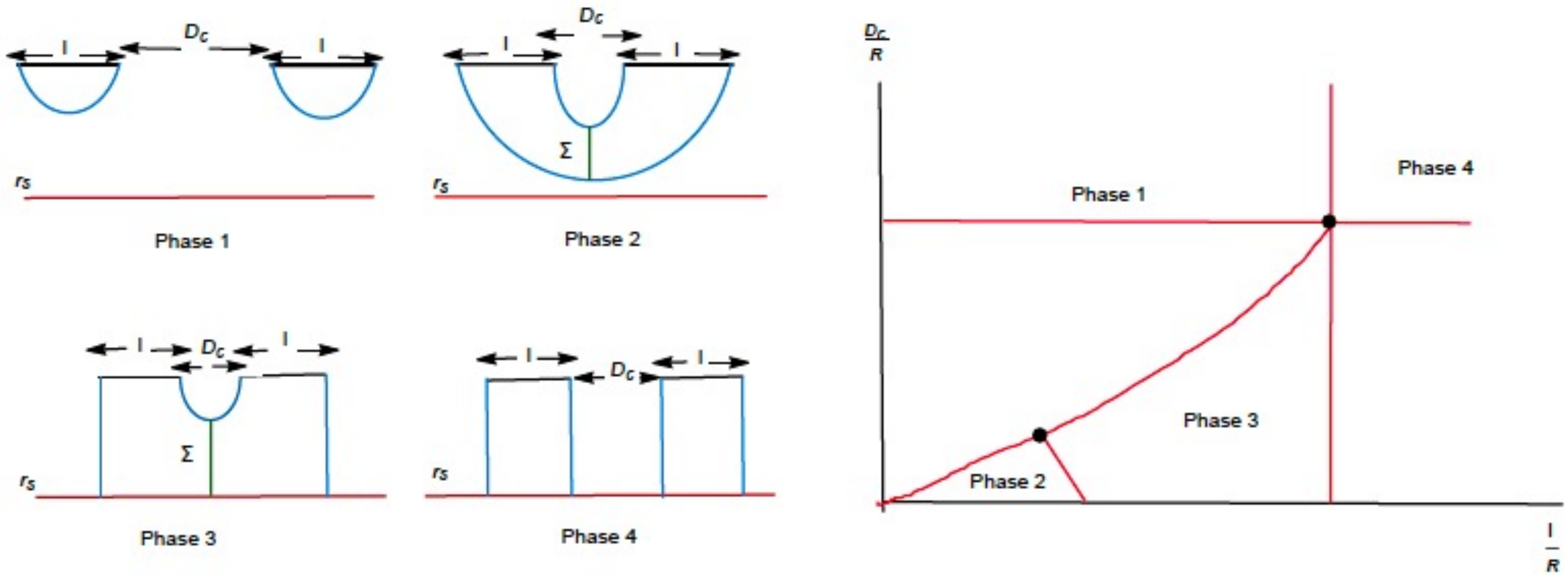
$$\Gamma = \int_{u_D}^{u_{2l+D}} du e^{-2\Phi} \sqrt{-\gamma_{ab}},$$

EW in Confining case

Our confining models

1- AdS-Soliton
2- Witten-Sakai-Sugimoto (and deformed Sakai-Sugimoto)
3- Witten-QCD
4- Klebanov-Strassler
5- Klebanov-Tseytlin
6- Klebanov-Witten
7- Maldacena-Nunez
8- Domain Wall AdS/QCD (D5-brane geometry)

Some Previous works on D_c in confining backgrounds



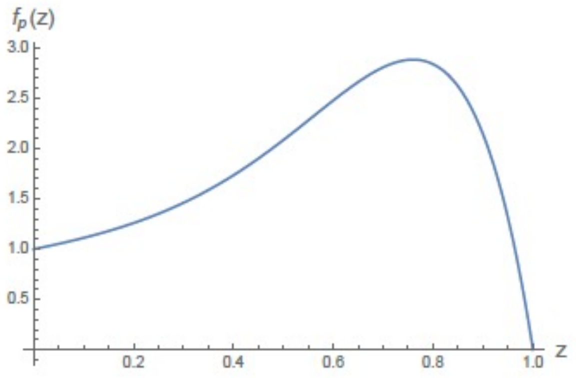
2010-07702 Jain-Mahapatra

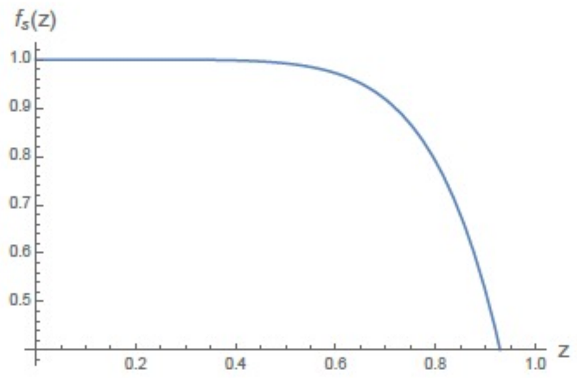
AdS-Soliton

AdS-soliton in Poincare gauge

$$ds^2 = \frac{L_{\text{AdS}}^2}{z^2}\left(\frac{dz^2}{f(z)} + f(z)d\theta^2 - dt^2 + dr^2 + r^2 d\Omega_{d-3}\right)$$

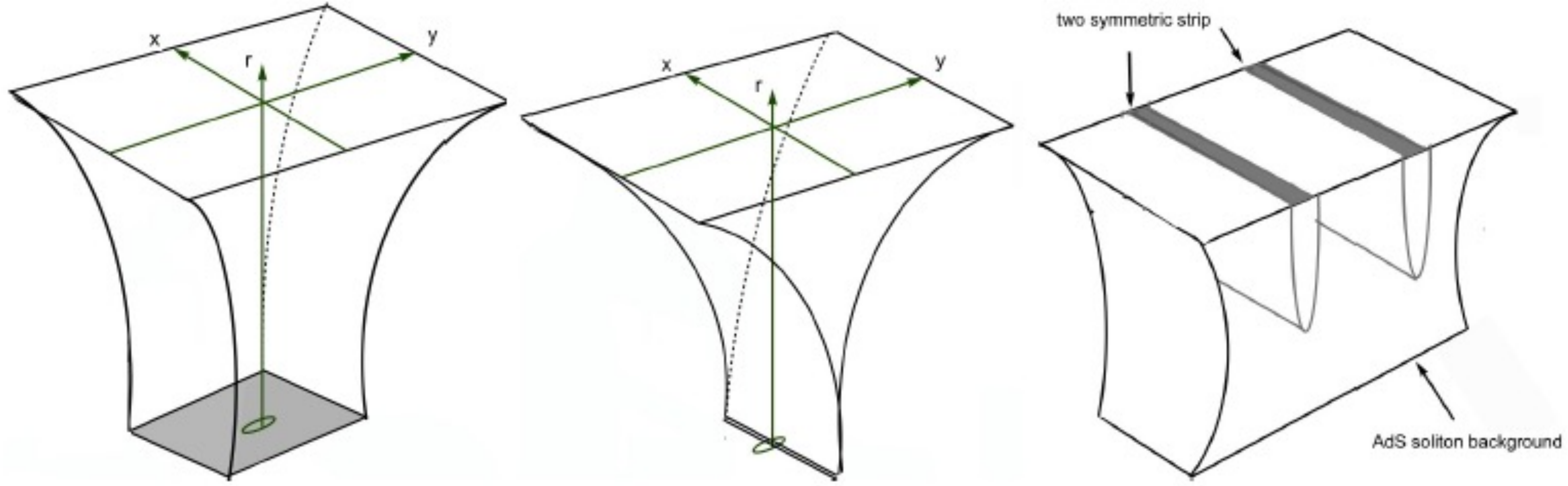
$$f(z) = \left(1 - k_1\frac{z}{z_0}\right)\left(1 - k_2\frac{z}{z_0}\right)(1 + \sum_{n=1} c_n z^n)$$





The behavior of $f_p(z)$ which uses the perturbative sum and arbitrary values of c_i ($k_1 = 1, k_2 = -2, c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 4, c_5 = 5$) and is used for the general case of soliton is shown in the left, and the function for the simplest AdS soliton case, $f_s(p) = 1 - (\frac{z}{z_0})^{8-d}, d = 1$ which we will use here is shown in the right to get intuitions about the metric of the AdS-soliton geometry.

Setup in AdS-Soliton



Other form of AdS-Soliton

General Asymptotic AdS background:

$$ds^2 = \frac{r^2}{\ell^2}\left[-\left(1-\frac{r_0^{p+1}}{r^{p+1}}\right)dt^2 + (dx^i)^2\right] + \left(1-\frac{r_0^{p+1}}{r^{p+1}}\right)^{-1}\frac{\ell^2}{r^2}dr^2,$$

$$E_p = \frac{pV_p}{16\pi G_{p+2}\ell^{p+2}}r_0^{p+1}$$

AdS-soliton can be derived by the analytical continuation of the above metric,

$t \to i\tau \qquad x^p \to it$

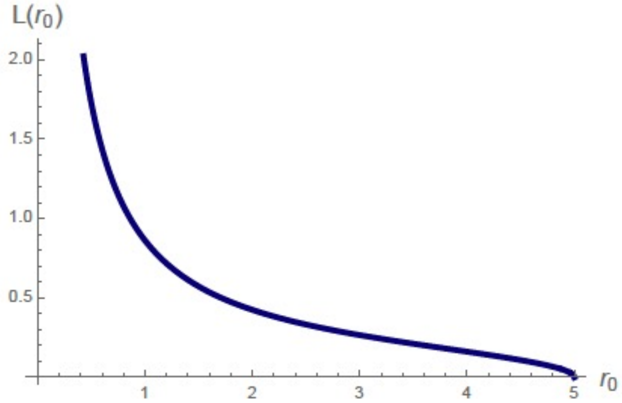
$$ds^2 = \frac{r^2}{\ell^2}\left[\left(1-\frac{r_0^{p+1}}{r^{p+1}}\right)d\tau^2 + (dx^i)^2 - dt^2\right] + \left(1-\frac{r_0^{p+1}}{r^{p+1}}\right)^{-1}\frac{\ell^2}{r^2}dr^2,$$

$$E = -\frac{r_0^{p+1}\beta V_{p-1}}{16\pi G_{p+2}\ell^{p+2}} \qquad \beta = 4\pi l^2/(p+1))r_0$$

Measures in AdS-Soliton

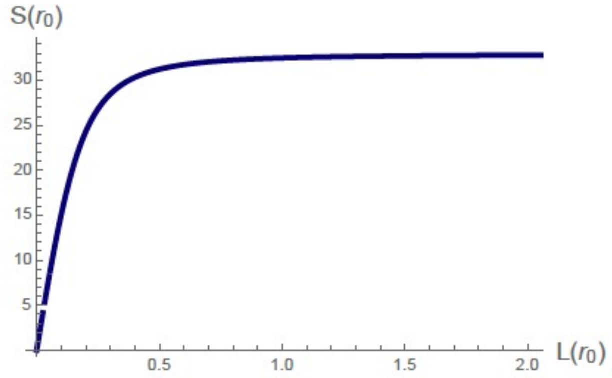
Width of the strip:

$$L(z_t)=\int_0^{z_t}\frac{2}{\sqrt{\left(1-\left(\frac{z}{z_0}\right)^{8-d}\right)\left(\frac{z_t^6}{z^6}\frac{1-\left(\frac{z_t}{z_0}\right)^{8-d}}{1-\left(\frac{z}{z_0}\right)^{8-d}}-1\right)}},$$



EE of the connected solution:

$$S_C(z_t)=\int_0^{z_t}\frac{c_1}{z^3\sqrt{1-\frac{z^6\left(1-\left(\frac{z}{z_0}\right)^{8-d}\right)}{z_t{}^6\left(1-\left(\frac{z_t}{z_0}\right)^{8-d}\right)}}},$$

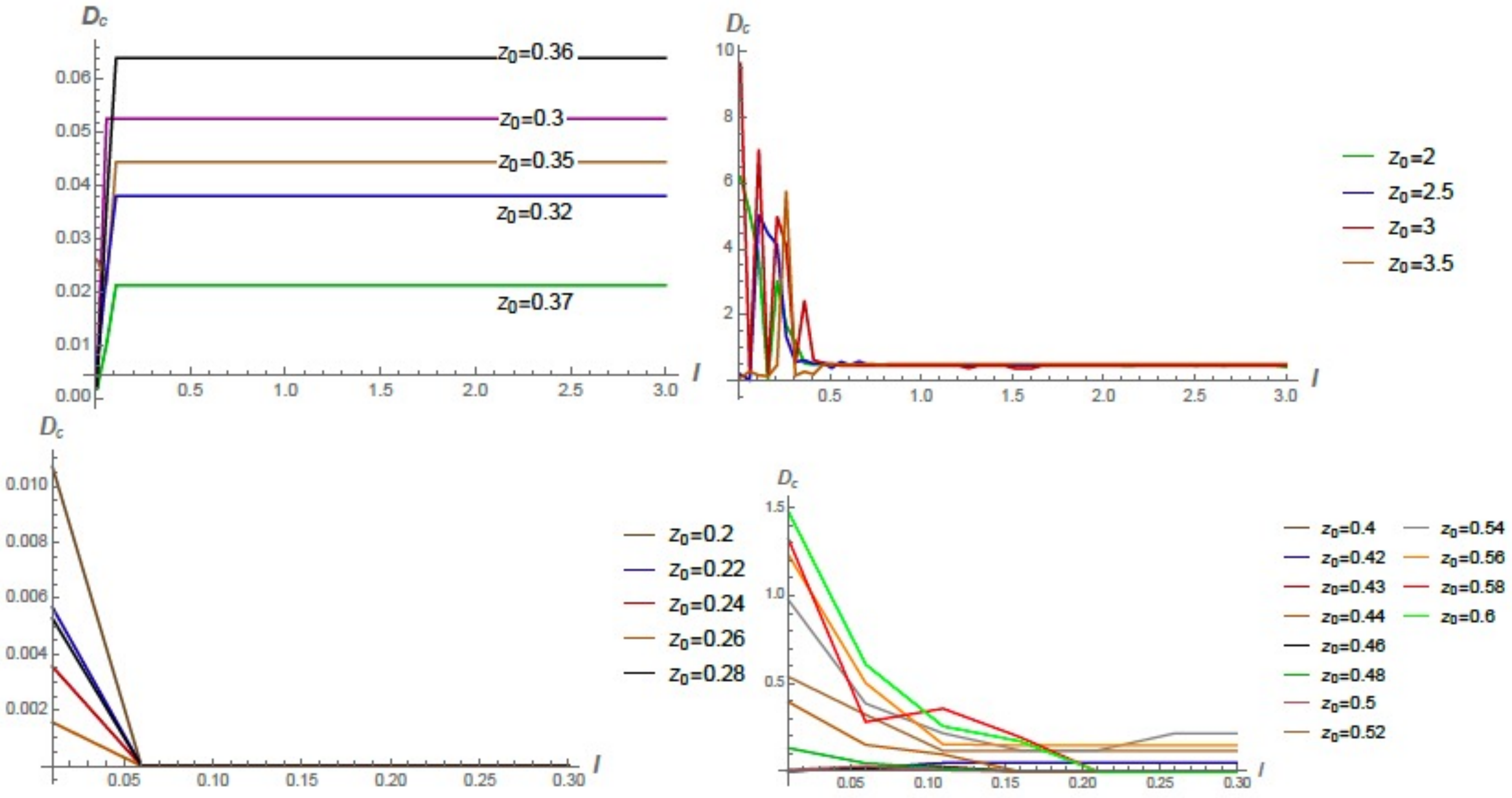


EE of the disconnected solution:

$$S_D=-\int_0^{z_0}\frac{c_1}{z^3},$$

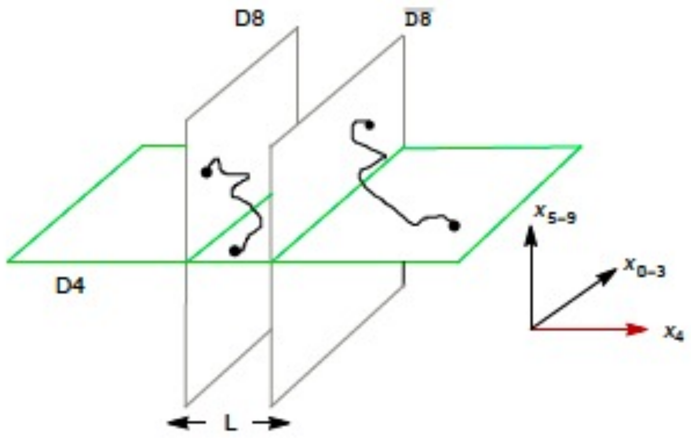
Here we mainly use S_C to construct the phase diagrams of D_c.

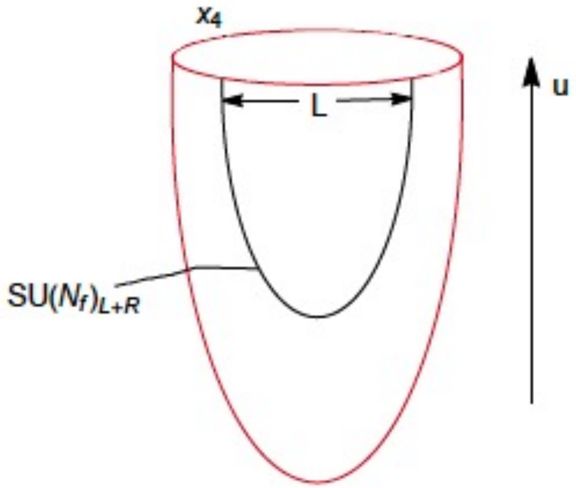
Phase diagrams of D_c in AdS-Soliton background



Witten-Sakai Sugimoto

Intersecting D4-D8 brane model of Sakai-Sugimoto. D8-branes are probe branes.



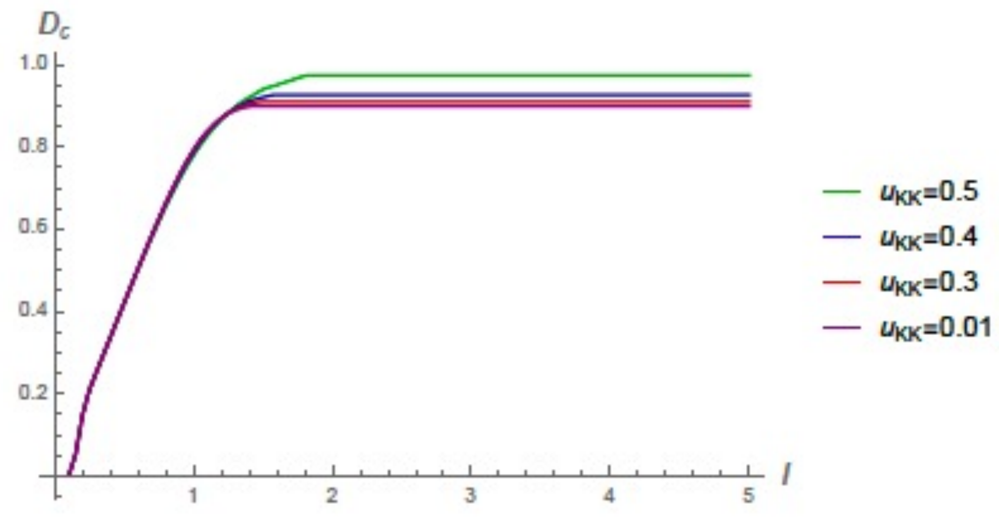


$$ds^2_{D4} = \left(\frac{u}{R_{D4}}\right)^{3/2}\left(-dt^2 + \delta_{ij}dx^i dx^j + f(u)(dx^4)^2\right) + \left(\frac{R_{D4}}{u}\right)^{3/2}\left(\frac{du^2}{f(u)} + u^2 d\Omega_4^2\right),$$

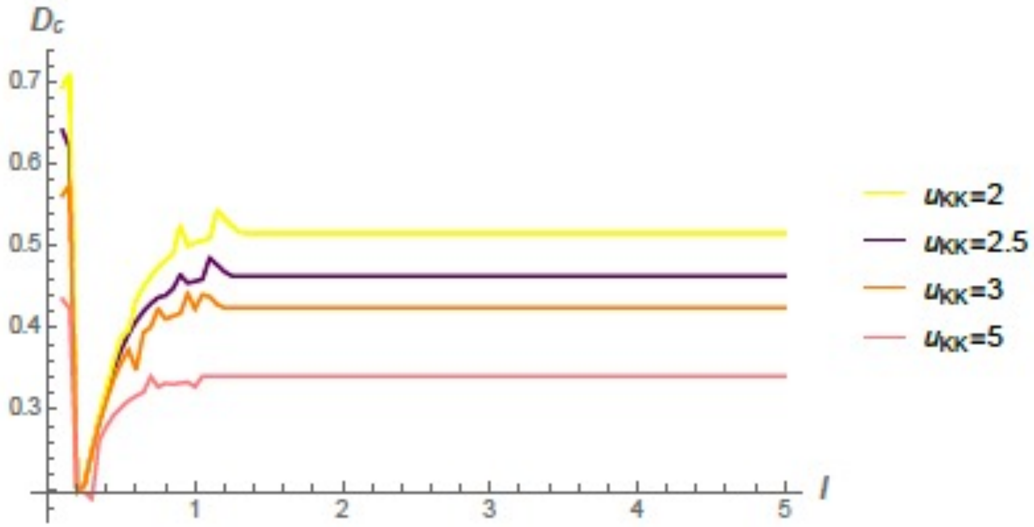
$$e^\phi = g_s\left(\frac{u}{R_{D4}}\right)^{3/4}, \quad F_4 \equiv dC_3 = \frac{2\pi N_c}{V_4}\epsilon_4, \quad f(u) \equiv 1 - \frac{u_{KK}^3}{u^3}, \quad R_{D4}^3 \equiv \pi g_s N_c l_s^3.$$

$$ds^2_{D8} = \left(\frac{u}{R_{D4}}\right)^{3/2}\left(-dt^2 + \delta_{ij}dx^i dx^j\right) + \left(\frac{R_{D4}}{u}\right)^{3/2}\left(\frac{du^2}{f(u)} + u^2 d\Omega_4^2\right)$$

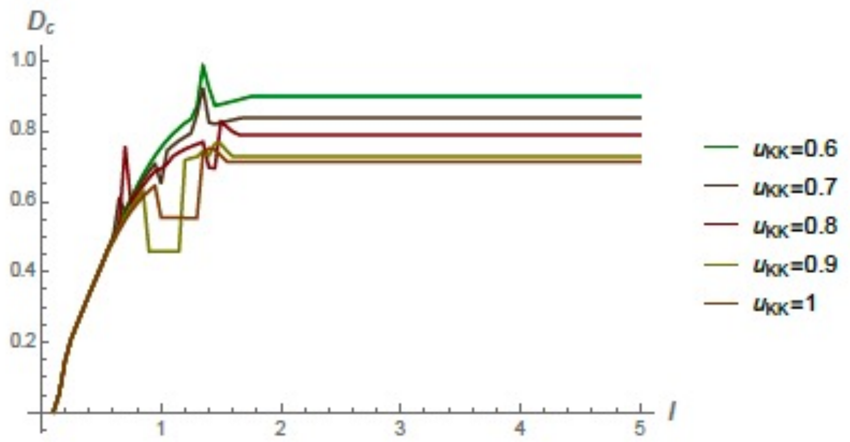
Critical distance in Sakai-Sugimoto model



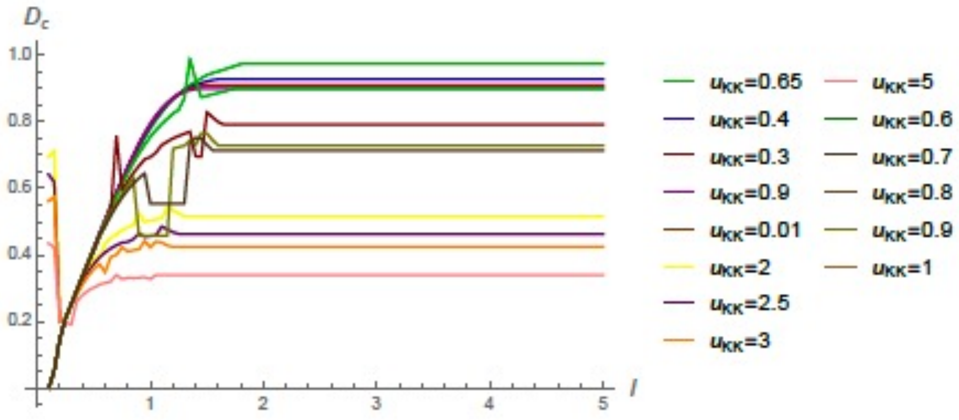
Phase 1



Phase 2

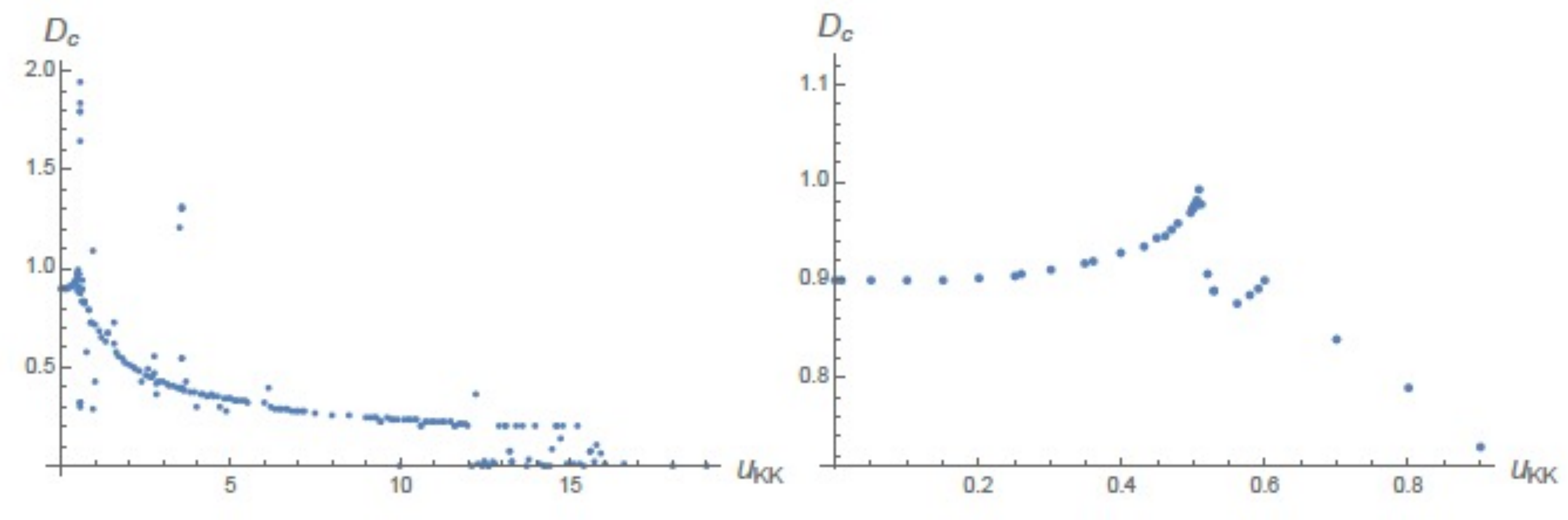


Phase 3



The three phases

The phases detected by D_c in the Sakai-Sugimoto background



In the left the whole four phases can be detected, while in the right, the first phase can be seen more clearly.

The first phase transition is related to chiral symmetry breaking/restoration, the second transition is related to confinement/deconfinement and the third is related to droping mutual information to zero or lowering its order of magnitude, leading to four different phases as we expected.

These phase transitions could be seen as the exchanges between different configurations of traversable wormholes, or specifically the Hawking-Page phase transition where:

In the confined phase

- Linear behavior of quark/anti-quark potential
- Low temperatures
- Finite values of $\sqrt{gtt.gxx}$
- The free energy of the order N_c^0

In the deconfined phase

- Decaying behavior of quark/anti-quark potential
- Higher temperatures
- Zero values of $\sqrt{gtt.gxx}$
- The free energy of the order N_c^2

When the quark separation is $L > L_c \cong 0.97*R$, the chiral symmetry would be restored and when $L < L_c \cong 0.97*R$ the chiral symmetry is broken in the deconfined phase and at $T_d = 1/2\,\pi\,R$. The chiral symmetry is restored again at $T_\chi \cong 0.154/L$.

The scale of chiral symmetry breaking/restoration is independent of confinement/deconfinement phase transition. The first one depend on the distance between D8-branes (or the mass scale of mesons, 1/L) in the model while the second one would depend on the radius where the D4-brane is wrapped around (or the mass scale of glueballs, 1/R).

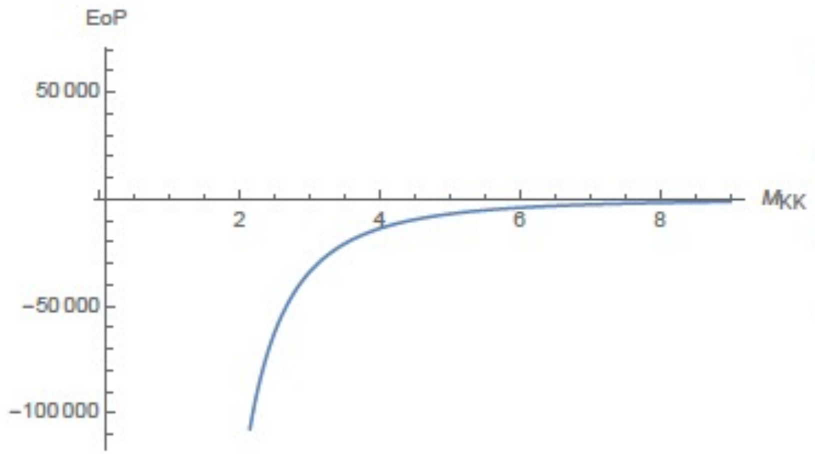
EW behavior in Sakai-Sugimoto geometry

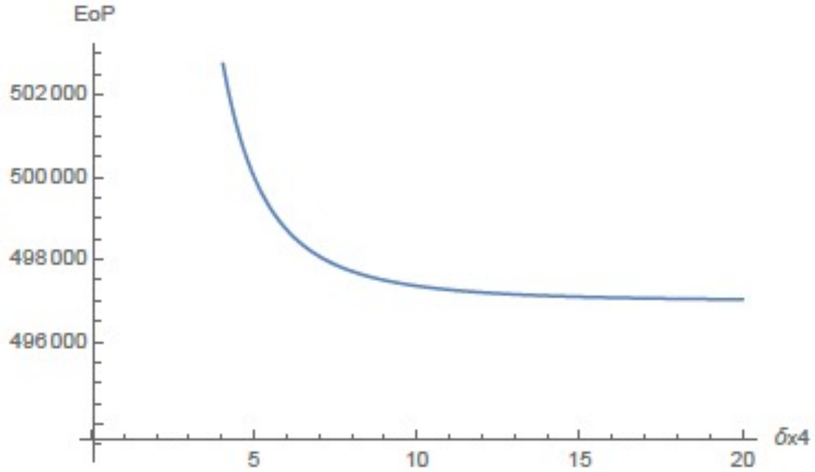
$$\boxed{\Gamma_{WSS} = R_{D4}^3 \int_{u_D}^{u_{2l+D}} du \frac{u^5}{1-\frac{u_{KK}^3}{u^3}}}$$

M_{KK}, g_{YM}, N_c are in the boundary

R_{D_4}, u_{KK}, g_s are in the bulk

$$R_{D_4}^3 = \frac{1}{2}\frac{\lambda l_s^2}{M_{KK}}, \quad u_{KK} = \frac{2}{9}\lambda M_{KK} l_s^2, \quad g_s = \frac{1}{2\pi}\frac{\lambda}{M_{KK} N_c l_s}$$

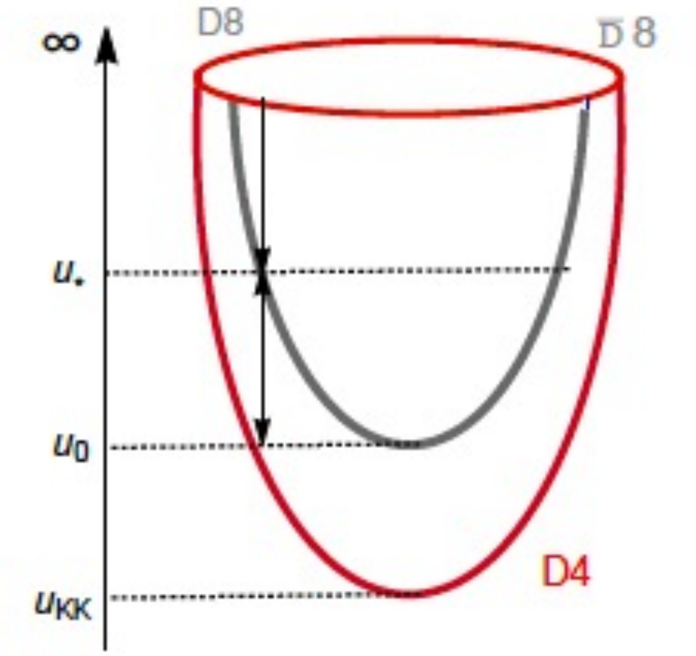




The string coupling g_s , the number of colors of gauge group N_c , the string length l_s and the cutoff M_{KK} can increase the EoP, while the periodicity of the boundary δx^4 would decrease EoP.

Deformed Sakai-Sugimoto

In this model x^4 instead of being constant, depends on the coordinate ”u”.



$$ds^2_{D8} = \left(\frac{u}{R_{D4}}\right)^{3/2}(-dt^2 + \delta_{ij}dx^i dx^j) + \left(\frac{u}{R_{D4}}\right)^{3/2}\frac{du^2}{h(u)} + \left(\frac{R_{D4}}{u}\right)^{3/2} u^2 d\Omega_4^2,$$

where

$$h(u) \equiv \left[f(u)\left(\frac{dx^4(u)}{du}\right)^2 + \left(\frac{R_{D4}}{u}\right)^3 \frac{1}{f(u)}\right]^{-1}$$

Measures in deformed Sakai-Sugimoto

$$S(u_t) = \frac{V_3 V_4 R_{D_4}^3}{2 g_s^2 G_N^{(10)}} \int_{u_t}^{\infty} du \frac{u}{\sqrt{(1-\frac{u_{KK}^3}{u^3})(1-\frac{u_t^5}{u^5})}} + \frac{V_3 V_4 R_{D_4}^{\frac{3}{2}}}{2 g_s^2 G_N^{(10)}} \int_{u_t}^{\infty} du \left(\frac{dx^4(u)}{du}\right) \sqrt{\frac{u^5(1-\frac{u_{KK}^3}{u^3})}{1-\frac{u_t^5}{u^5}}}.$$

The second term is the difference with Sakai-Sugimoto case. $dx^4(u)/du$ controls the difference.

$$L(u_t) = 2 R_{D_4}^{\frac{3}{2}} \int_{u_t}^{\infty} du \frac{1}{\sqrt{u^3\left(1-\frac{u_{KK}^3}{u^3}\right)\left(\frac{u^5}{u_t^5}-1\right)}} + 2 \int_{u_t}^{\infty} du \left(\frac{dx^4(u)}{du}\right) \sqrt{\frac{1-\frac{u_{KK}^3}{u^3}}{\frac{u^5}{u_t^5}-1}}$$

The second term is the difference with Sakai-Sugimoto case. $dx^4(u)/du$ controls the difference.

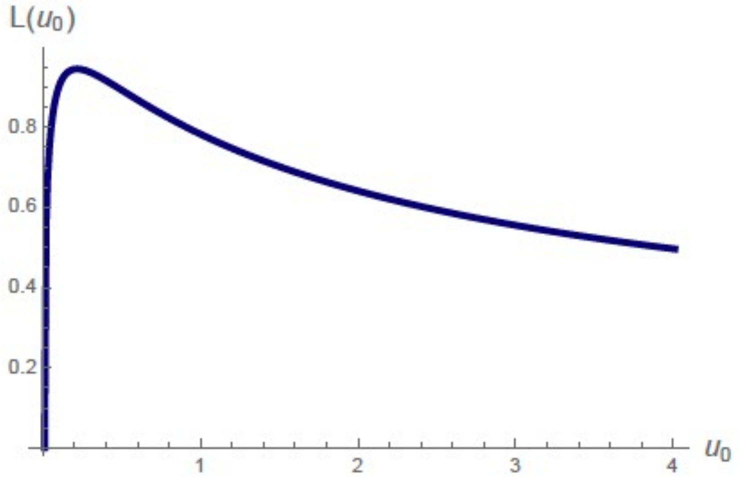
$$\Gamma_{\text{Deformed SS}} = R_{D4}^3 \int_{u_D}^{u_{2l+D}} du \frac{u^5}{1-\frac{u_{KK}^3}{u^3}} + \int_{u_D}^{u_{2l+D}} du u^8 \left(1-\frac{u_{KK}^3}{u^3}\right) \left(\frac{dx^4(u)}{du}\right)^2$$

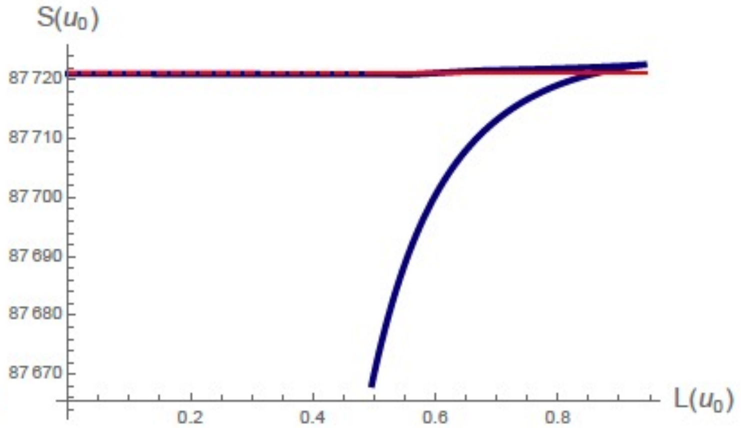
The second term is the difference with Sakai-Sugimoto case. $dx^4(u)/du$ controls the difference.

Witten-QCD

$$ds^2 = (\frac{u}{R})^{3/2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + \frac{4R^3}{9u_t}f(u)d\theta^2\right) + \left(\frac{R}{u}\right)^{3/2}\frac{du^2}{f(u)} + R^{3/2}u^{1/2}d\Omega_4^2,$$

$$f(u) = 1 - \frac{u_t^3}{u^3}, \qquad R = (\pi N g_s)^{\frac{1}{3}}\alpha'^{\frac{1}{2}}, \qquad e^\Phi = g_s\frac{u^{3/4}}{R^{3/4}}.$$

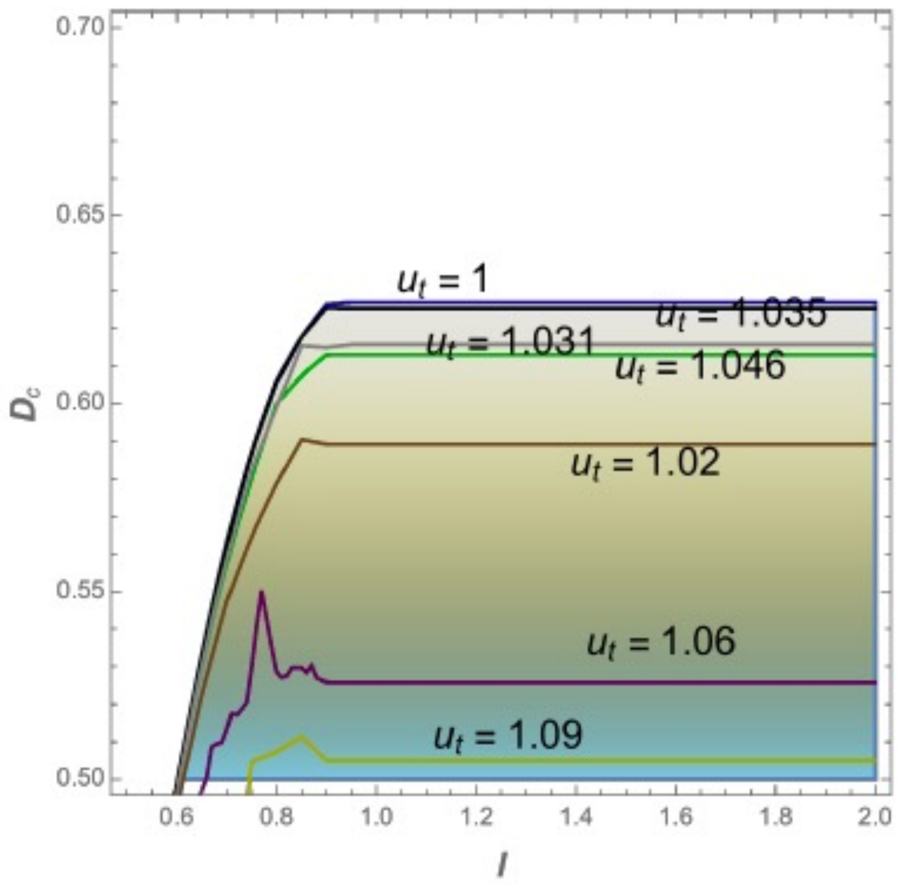


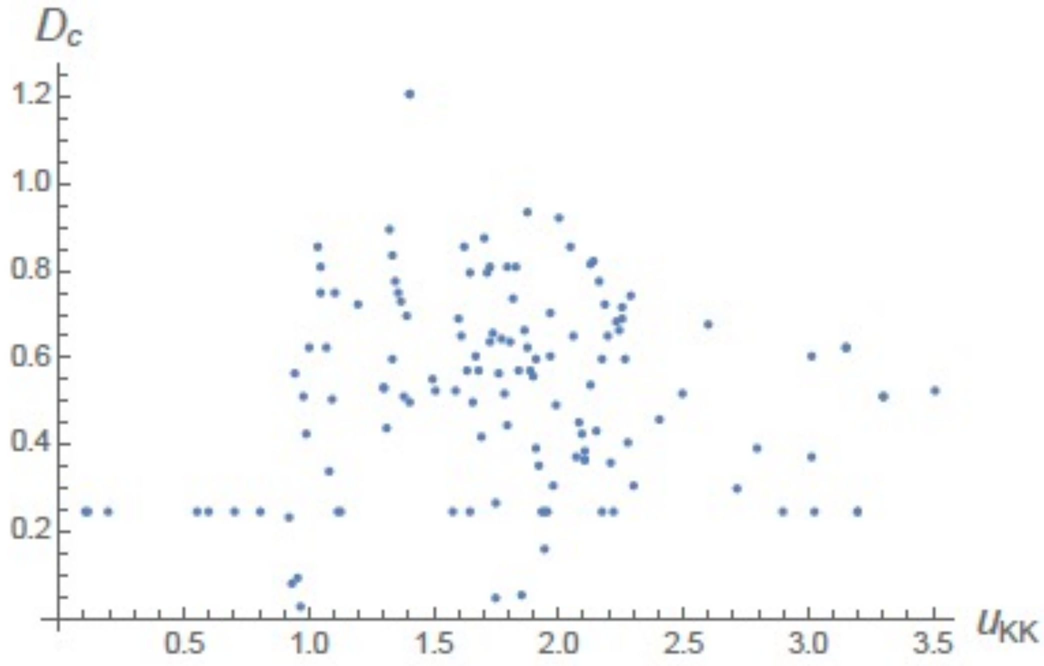


$$L(u_0) = 2\int_{u_0}^{\infty} du\sqrt{\frac{(\frac{R}{u})^3\frac{1}{f(u)}}{\frac{f(u)u^5}{f(u_0)u_0^5} - 1}},$$

$$S_C(u_0) = \frac{V_2}{G_N^{(10)}}\frac{8\pi^2 R^{\frac{9}{2}}}{9g_s^2\sqrt{u_t}}\int_{u_0}^{\infty} du\sqrt{\frac{u^2}{1 - \frac{u_0^5}{u^5}.\frac{f(u_0)}{f(u)}}}$$

$$\Gamma_{\rm WQCD} = \int_{u_D}^{u_{2l+D}} du\frac{2R^{9/2}u}{3u_0^{1/2}g_s^2},$$





Again three phase transitions related to chiral symmetry breaking/restoration, confinement/deconfinement and change of order of MI could be seen. The first one around u_{KK}-=1, the second around u_{KK}-=1.3 and the third around u_{KK}-=1.7, and then D_c would become zero by increasing u_{KK}. Note that quarks are considered massless in this model which make chiral symmetry breaking a “sharp” jump, as it is like an order parameter, the same for deconfinement which breaks the Z_{Nc} global symmetry.

Sources of the noises?

1-Numerical errors.

2- In all the holographic models of QCD, regardless of the metric, a Chern-Simons coupling would exist which couples the vector and axial-vector mesons and mix the transverse polarizarion states which then can produce instabilities which could produce these noises.

3- Interplay between thiscondensation and other effects, such as pion condensation, chiral symmetry breaking, colour superconductivity, mixed polarization states, etc,

Klebanov-Strassler

- Type-IIB supergravity solution which has various fluxes and warp factors and are dual to confining $\mathcal{N}$=1 SYM theories which describe the barionic branch of dual gauge theory.

- It is constructed by the collection of N regular and M fractional D3-branes in the geometry of deformed conifold. The fractional D3-branes could be thought as D5-branes that are wraped the two-cycle of $T^{1,1}$ which collapse at the apex τ=0.

- There are two free parameters in KS which their dual are baryonic VEV and the gauge coupling constant. The Maldacena-Nunez solution could also be considered as the end point of a flow from KS by changing these two parameters.

- The gauge group on the stack of N D3-branes and M fractional D3-branes would be SU(N +M)* SU(N). The limit where M << N would correspond to the Klebanov-Tseytlin solution

Klebanov-Strassler

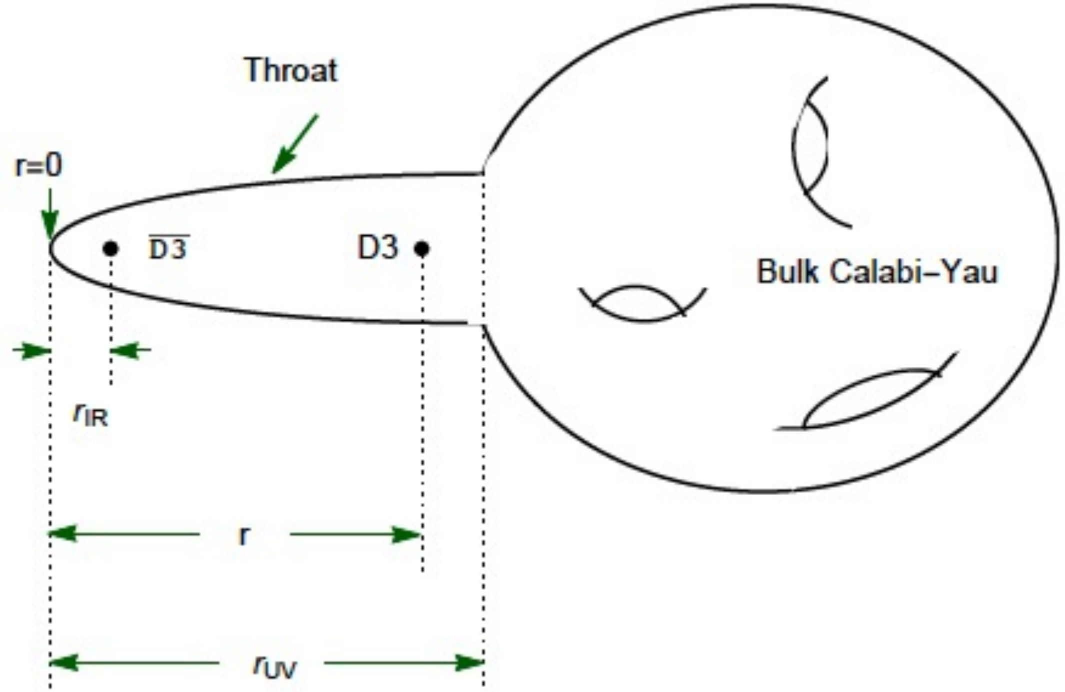
Metric

$$ds_{10}^2 = h^{-\frac{1}{2}}(\tau)dx_\mu dx^\mu + h^{\frac{1}{2}}(\tau)ds_6{}^2,$$

$$ds_6{}^2 = \frac{1}{2}\epsilon^{\frac{4}{3}}K(\tau)\Big[\frac{1}{3K^3(\tau)}(d\tau^2 + (g^5)^2) + \cosh^2(\frac{\tau}{2})[(g^3)^2 + (g^4)^2] + \sinh^2(\frac{\tau}{2})[(g^1)^2 + (g^2)^2]\Big]$$

K is a decreasing function of the radial coordinate:

$$K(\tau) = \frac{(\sinh 2\tau - 2\tau)^{\frac{1}{3}}}{2^{\frac{1}{3}} \sinh \tau}$$



$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{\frac{1}{3}}}{2^{\frac{1}{3}} \sinh \tau}, \qquad h(\tau) = (g_s M\alpha')^2 2^{2/3} \epsilon^{-8/3} I(\tau),$$

$$I(\tau) = \int_\tau^\infty dx \frac{x \coth x - 1}{\sinh^2 x}(\sinh(2x) - 2x)^{\frac{1}{3}},$$

Klebanov-Strassler

$$\boxed{ds_{10}^2 = h^{-\frac{1}{2}}(\tau)dx_\mu dx^\mu + h^{\frac{1}{2}}(\tau)ds_6{}^2,}$$

$$\boxed{ds_6{}^2 = \frac{1}{2}\epsilon^{\frac{4}{3}}K(\tau)\Big[\frac{1}{3K^3(\tau)}(d\tau^2 + (g^5)^2) + \cosh^2(\frac{\tau}{2})[(g^3)^2 + (g^4)^2] + \sinh^2(\frac{\tau}{2})[(g^1)^2 + (g^2)^2]\Big]}$$

$$g^1 = \frac{1}{\sqrt{2}}[-\sin\theta_1 d\phi_1 - \cos\psi\sin\theta_2 d\phi_2 + \sin\psi d\theta_2], \qquad g^2 = \frac{1}{\sqrt{2}}[d\theta_1 - \sin\psi\sin\theta_2 d\phi_2 - \cos\psi d\theta_2],$$

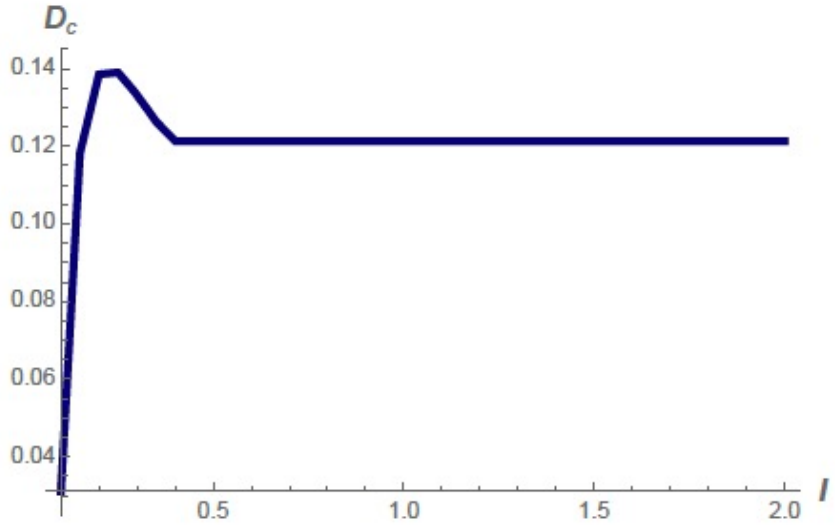
$$g^3 = \frac{1}{\sqrt{2}}[-\sin\theta_1 d\phi_1 + \cos\psi\sin\theta_2 d\phi_2 - \sin\psi d\theta_2], \qquad g^4 = \frac{1}{\sqrt{2}}[d\theta_1 + \sin\psi\sin\theta_2 d\phi_2 + \cos\psi d\theta_2],$$

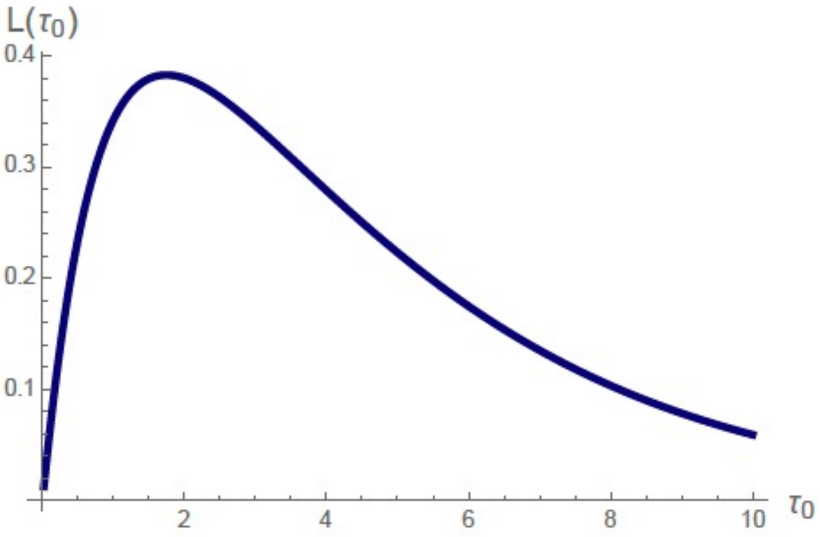
$$g^5 = d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2.$$

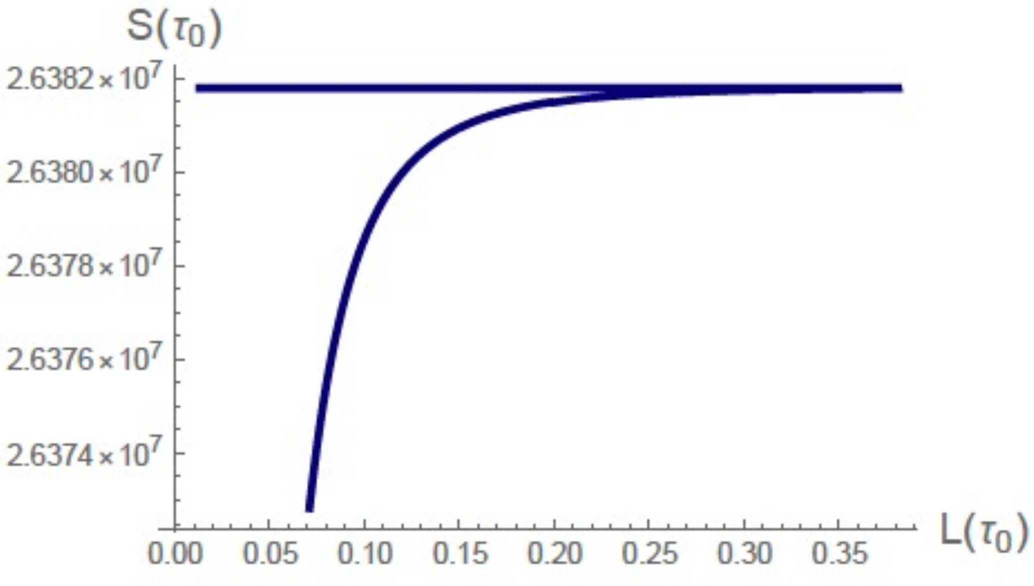
Strips in KS

$$L(\tau_0) = \frac{2^{\frac{5}{6}}\epsilon^{\frac{2}{3}}}{\sqrt{3}} \int_{\tau_0}^{\infty} d\tau \frac{\sinh(\tau)}{(\sinh(2\tau) - 2\tau)^{\frac{1}{3}}} \sqrt{\frac{h(\tau)}{\frac{\sinh^2(\tau)}{\sinh^2(\tau_0)} \left(\frac{\sinh(2\tau)-2\tau}{\sinh(2\tau_0)-2\tau_0}\right)^{\frac{2}{3}} - 1}},$$

$$S_C(\tau_0) = \frac{V_2 \pi^3 \epsilon^4}{3G_N^{(10)}} \int_{\tau_0}^{\infty} d\tau \frac{h(\tau)\sinh^2(\tau)}{\sqrt{1 - \left(\frac{\sinh \tau_0}{\sinh \tau}\right)^2 \left(\frac{\sinh(2\tau_0)-2\tau_0}{\sinh(2\tau)-2\tau}\right)^{\frac{2}{3}}}}.$$







EW in KS

$$\det(\gamma_{ab}) = \frac{h^4 \epsilon^8}{9216} \sin^2\theta_1 \sin^2\theta_2 \left(\cos^2\psi - \cosh^2\left(\frac{\tau}{2}\right)\cosh\tau\right)\left(\cos^2\psi - \cosh^2\tau\right)$$

$$\boxed{\Gamma_{\mathrm{KS}} = \frac{\epsilon^8}{9216}\int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^{2\pi} d\psi \int_{\tau_D}^{\tau_{2l+D}} d\tau \; h^4(\tau)\left(\cos^2\psi - \cosh^2\left(\frac{\tau}{2}\right)\cosh\tau\right)\left(\cos^2\psi - \cosh^2\tau\right)}$$

Klebanov-Tseytlin

The Klebanov-Tseytin metric is a singular solution (so the position of singularity can be changes as the free parameter).
It is dual to the chirally-symmetric phase of the Klebanov-Strassler model which has D3-brane charges that dissolve in the flux.

$$ds_{10}^2 = h(r)^{-1/2}\left[-dt^2 + d\vec{x}^2\right] + h(r)^{1/2}\left[dr^2 + r^2 ds_{T^{1,1}}^2\right]$$

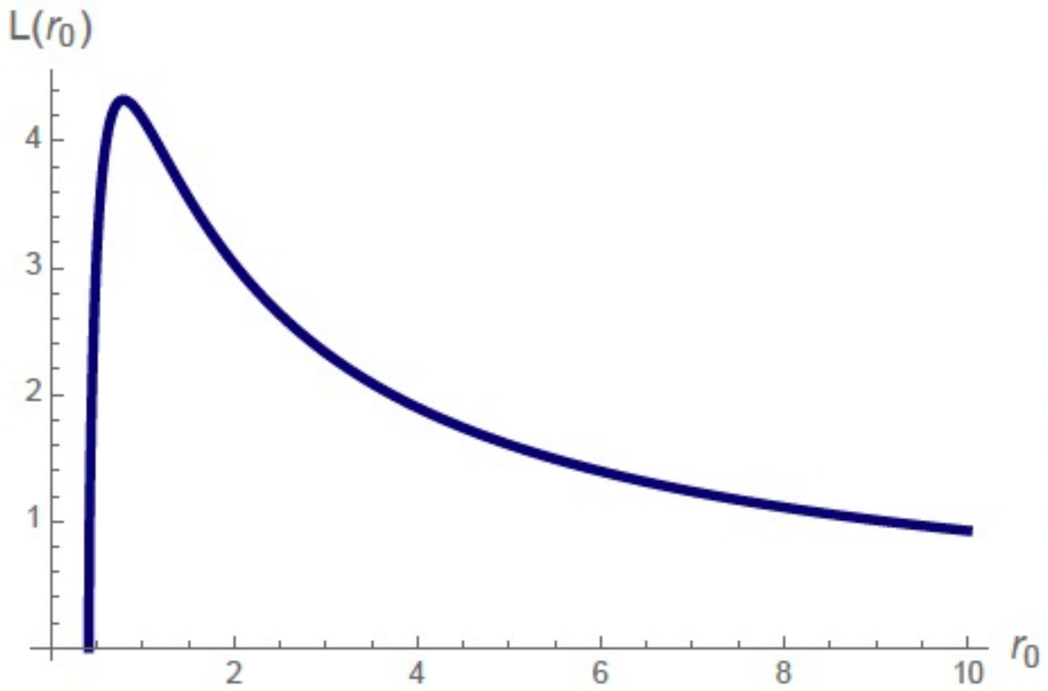
$$\boxed{ds_{T^{1,1}}^2 = \frac{1}{9}(g^5)^2 + \frac{1}{6}\sum_{i=1}^{4}(g^i)^2}$$

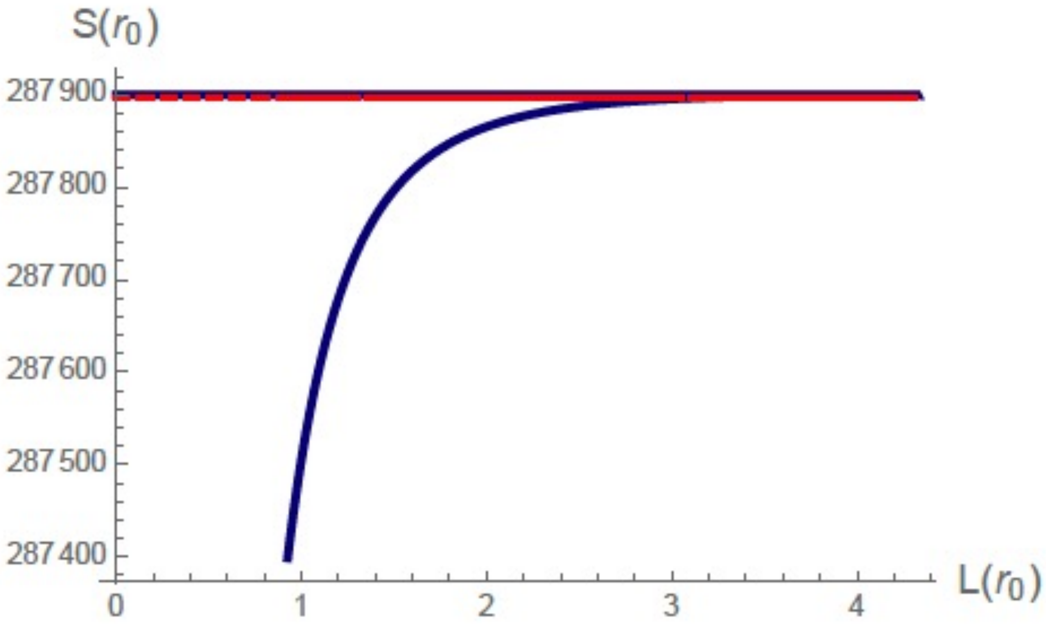
$$g^1 = (-\sin\theta_1 d\phi_1 - \cos\psi \sin\theta_2 d\phi_2 + \sin\psi d\theta_2)/\sqrt{2}, \qquad g^2 = (d\theta_1 - \sin\psi\sin\theta_2 d\phi_2 - \cos\psi d\theta_2)/\sqrt{2},$$
$$g^3 = (-\sin\theta_1 d\phi_1 + \cos\psi\sin\theta_2 d\phi_2 - \sin\psi d\theta_2)\sqrt{2}, \qquad g^4 = (d\theta_1 + \sin\psi\sin\theta_2 d\phi_2 + \cos\psi d\theta_2)/\sqrt{2},$$

$$g^5 = d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2$$

$$\boxed{h(r) = \frac{L^4}{r^4}\ln\frac{r}{r_s}, \qquad L^4 = \frac{81}{2} g_s M^2 \epsilon^4.}$$

Strips in KT

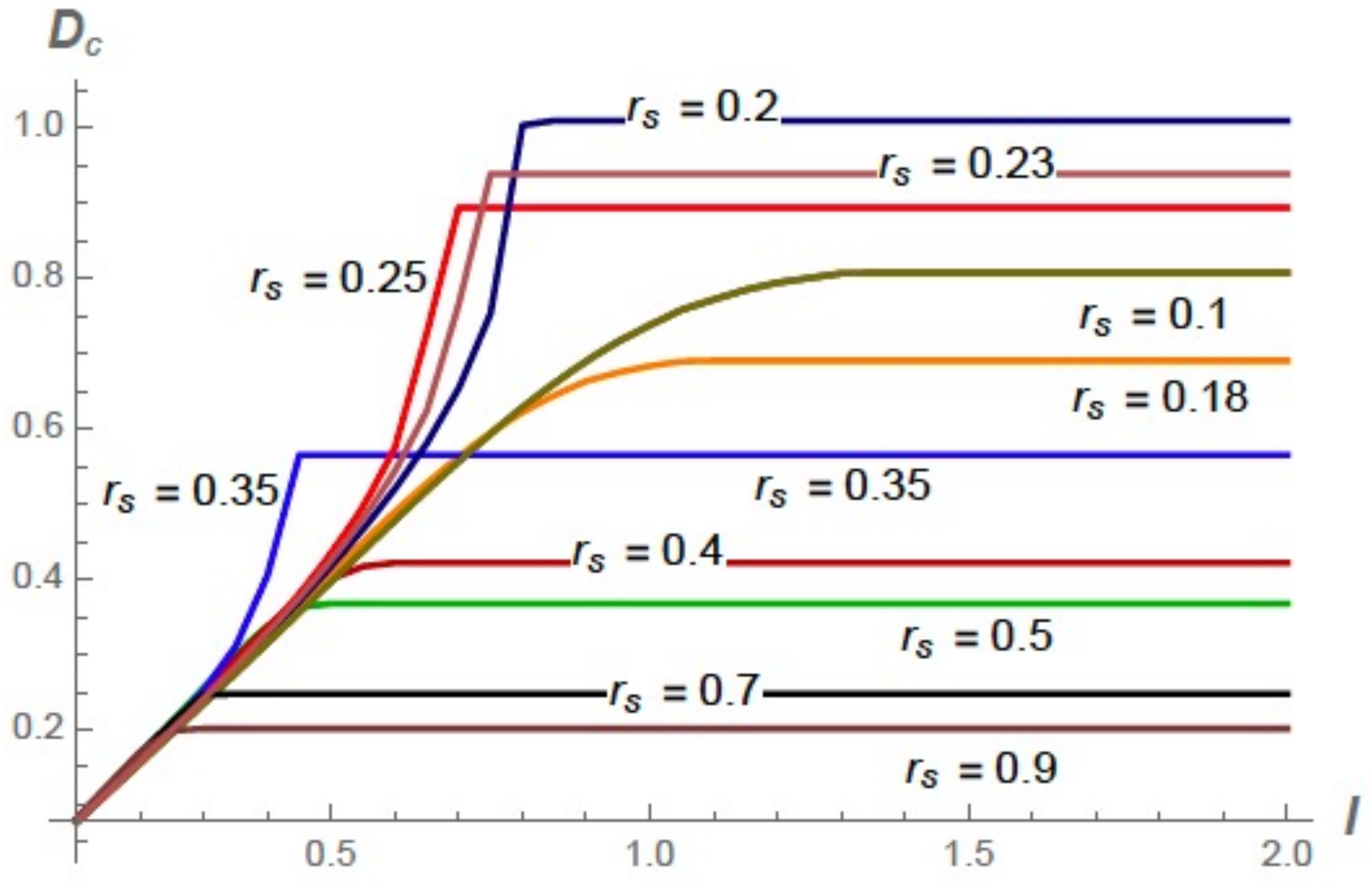




$$L(r_0) = 9\sqrt{2}M\sqrt{g_s}\epsilon^2 \int_{r_0}^{\infty} dr \frac{\sqrt{\ln\frac{r}{r_s}}}{r^2\sqrt{\frac{r^6}{r_0^6}\frac{\ln\frac{r}{r_s}}{\ln\frac{r_0}{r_s}} - 1}},$$

$$S_C(r_0) = \frac{12V_2\pi^3M^2g_s\epsilon^4}{G_N^{(10)}} \int_{r_0}^{\infty} dr \frac{r\ln\frac{r}{r_s}}{\sqrt{1 - \frac{r_0^6}{r^6}\frac{\ln\frac{r_0}{r_s}}{\ln\frac{r}{r_s}}}},$$

Phase structures from D_c in KT



Again, four different phases could be detected, and three phase transitions.

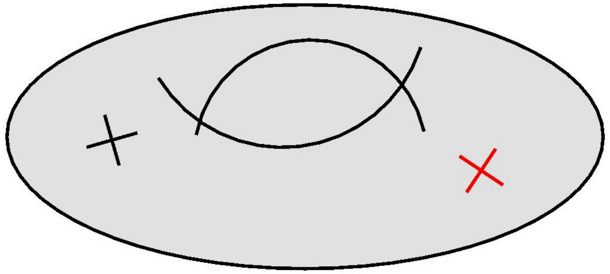
Klebanov-Witten

The Klebanov-Witten solution is similar to the KT throat geometry but with no logarithmic warping. Unlike the other four mentioned metrics, it is just ***conformal*** geometry.

metric

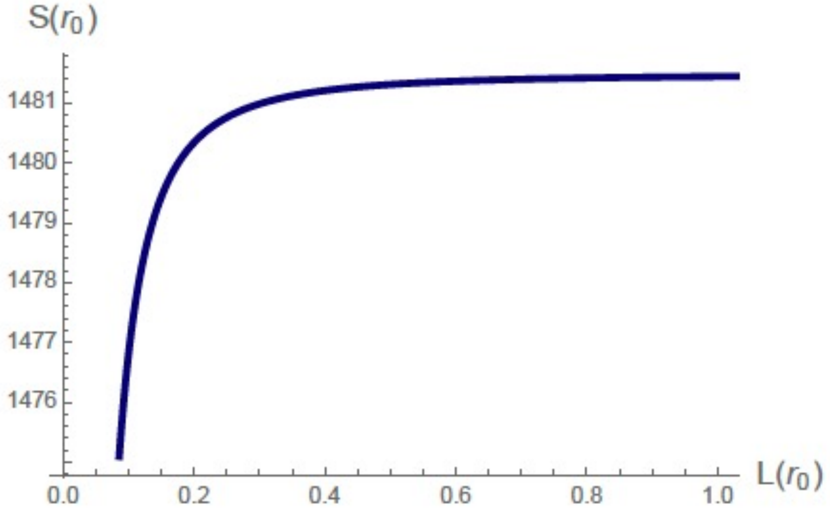
$$ds^2 = h^{-\frac{1}{2}} g_{\mu\nu} dx^\mu dx^\nu + h^{\frac{1}{2}} (dr^2 + r^2 ds^2_{T^{1,1}})$$

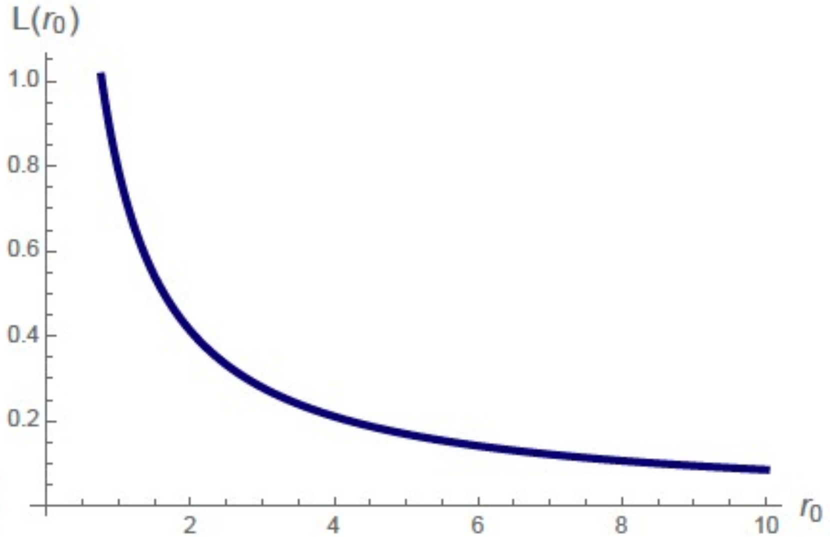
$$h = \frac{L^4}{r^4}, \qquad \text{and} \qquad L^4 = \frac{27\pi}{4} g_s N (\alpha')^2$$



Klebanov–Witten

Strips in KW





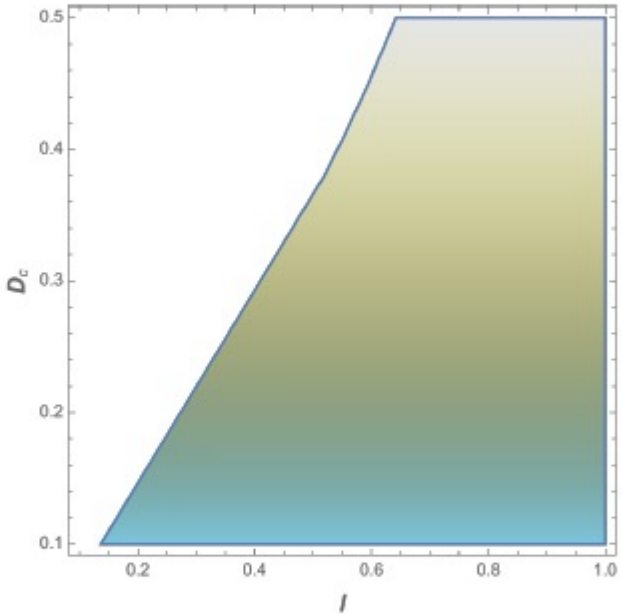
Similar to AdS case as both are conformal.

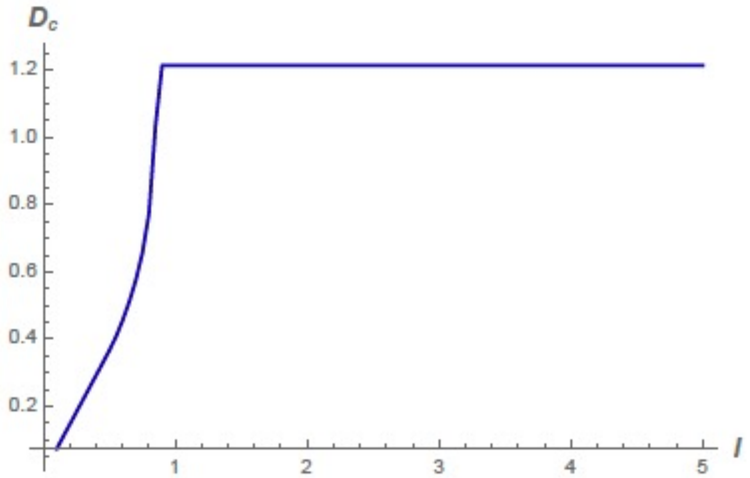
$$L(r_0) = 2L^2 \int_{r_0}^{\infty} dr \frac{1}{r^2\sqrt{\frac{r^6}{r_0^6} - 1}},$$

$$S(r_0) = \frac{8V_{d-1}\pi^4 L}{27G_N^{(10)}} \int_{r_0}^{\infty} dr \frac{r}{\sqrt{1 - \frac{r_0^6}{r^6}}}.$$

Mixed measures in KW

$$\Gamma_{\text{KW}} = \frac{1}{108\sqrt{6}} \int_{r_D}^{r_{2l+D}} dr h^2(r) r^5 \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \sqrt{-3 + \cos 2\theta_1 + 2\cos 2\theta_2} \sin\theta_1$$





The behavior of D_c is also very similar to BTZ case.

Maldacena-Nunez

The Maldacena-Nunez (MN) metric is obtained by a large number of D5-branes wrapping on S^2.

The Maldacena-Nunez solution which could also be considered as the end point of a flow from KS by changing the two parameters of dual to **baryonic VEV** and the **gauge coupling constant** in the boundary.

$$ds_{10}^2 = e^{\phi}[-dt^2 + dx_1{}^2 + dx_2{}^2 + dx_3{}^2 + e^{2h(r)}(d\theta_1{}^2 + \sin^2\theta_1 d\phi_1{}^2) + dr^2 + \frac{1}{4}(w^i - A^i)^2],$$

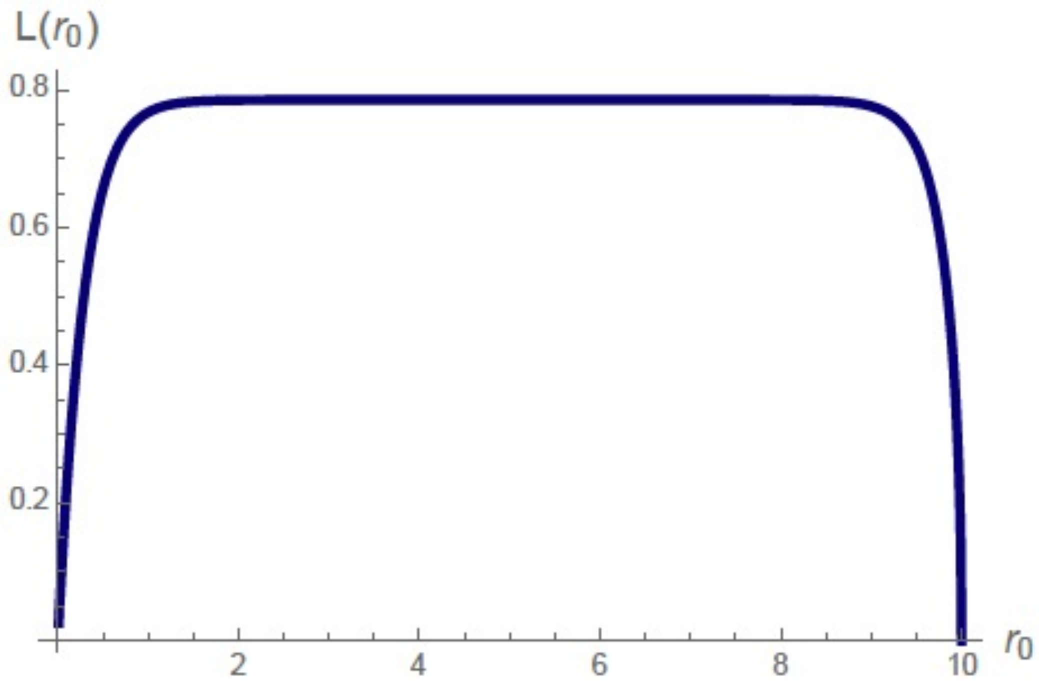
$$A^1 = -a(r)d\theta_1, \qquad A^2 = a(r)\sin\theta_1 d\phi_1, \qquad A^3 = -\cos\theta_1 d\phi_1,$$

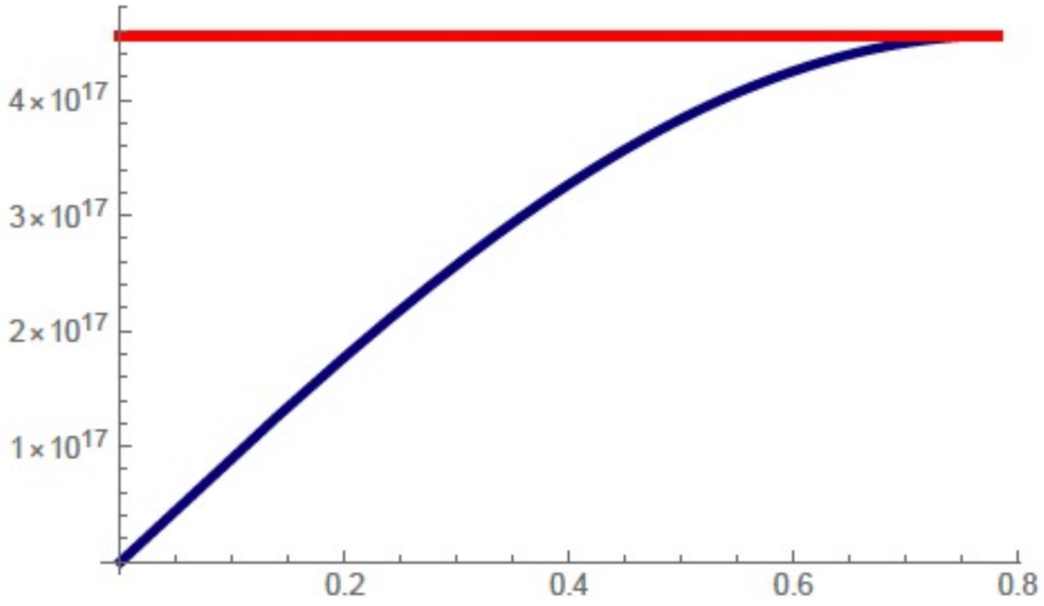
$$w^1 = \cos\psi d\theta_2 + \sin\psi\sin\theta_2 d\phi_2, \qquad w^2 = -\sin\psi d\theta_2 + \cos\psi\sin\theta_2 d\phi_2,$$
$$w^3 = d\psi + \cos\theta_2 d\phi_2,$$

Metric and Parameters:

$$a(r) = \frac{2r}{\sinh 2r}, \qquad e^{2h} = r\coth 2r - \frac{r^2}{\sinh 2r^2} - \frac{1}{4}, \qquad e^{-2\phi} = e^{-2\phi_0}\frac{2e^h}{\sinh 2r}.$$

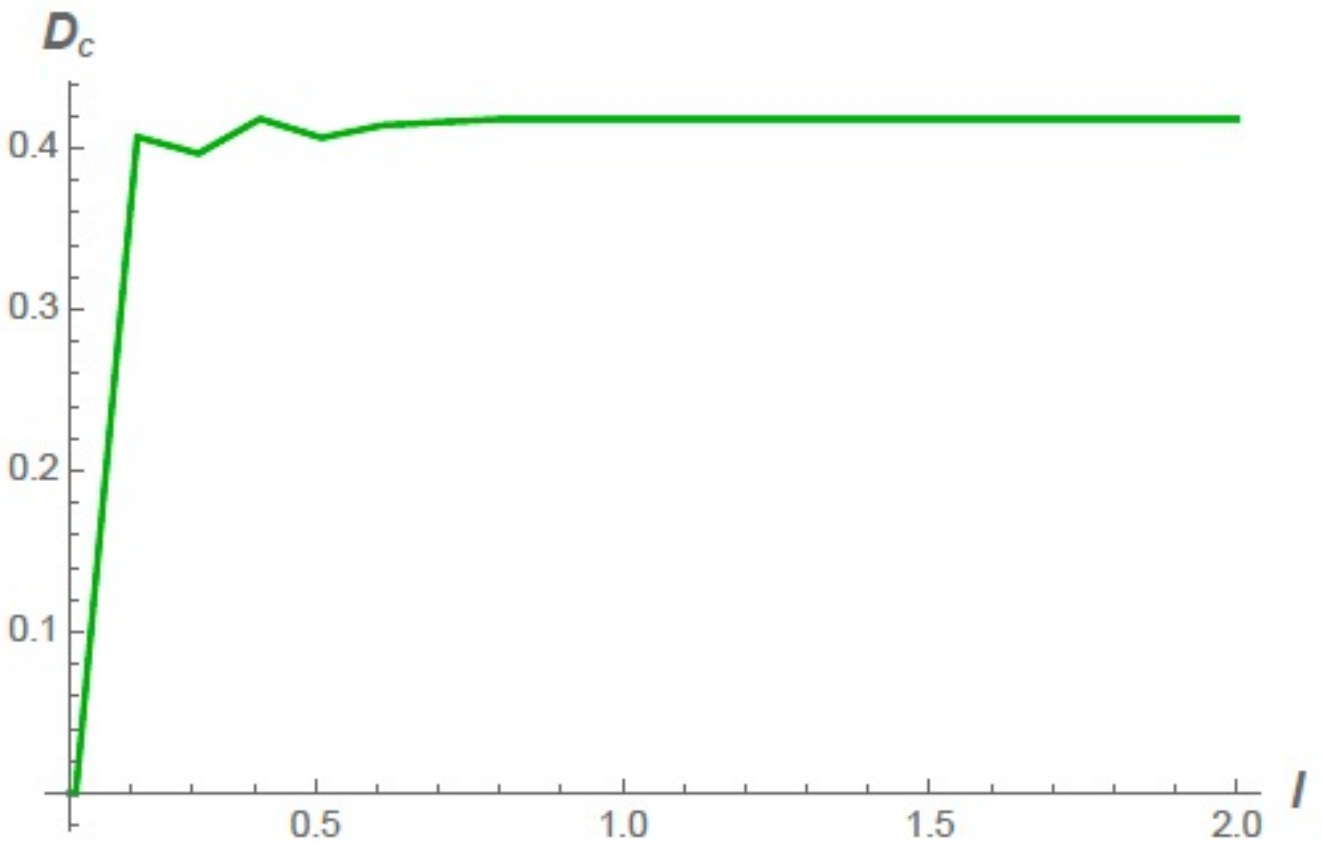
Strips in MN





$$L(r_0) = \int_{r_0}^{\infty} dr \frac{2}{\sqrt{\frac{\sinh^4(2r)}{\sinh^4(2r_0)} - 1}}, \qquad S_C(r_0) = \frac{V_2 \pi^3 e^{4\phi_0}}{G_N^{(10)}} \int_{r_0}^{\infty} dr \frac{\sinh^2(2r)}{\sqrt{1 - \frac{\sinh^4(2r_0)}{\sinh^4(2r)}}}.$$

Mixed measures in MN



$$\det\gamma_{ab} = e^{4h+8\phi}\sin^2(\theta_1)\sin^2(\theta_2)$$

$$\boxed{\Gamma_{\mathrm{MN}} = \int_{r_D}^{r_{2l+D}} dr e^{4\phi_0}\cosh^2(r)\sinh^2(r).}$$

The only free parameter is ϕ_0 . No rich phase structure could be detected in this case.

Domain-Wall AdS/QCD

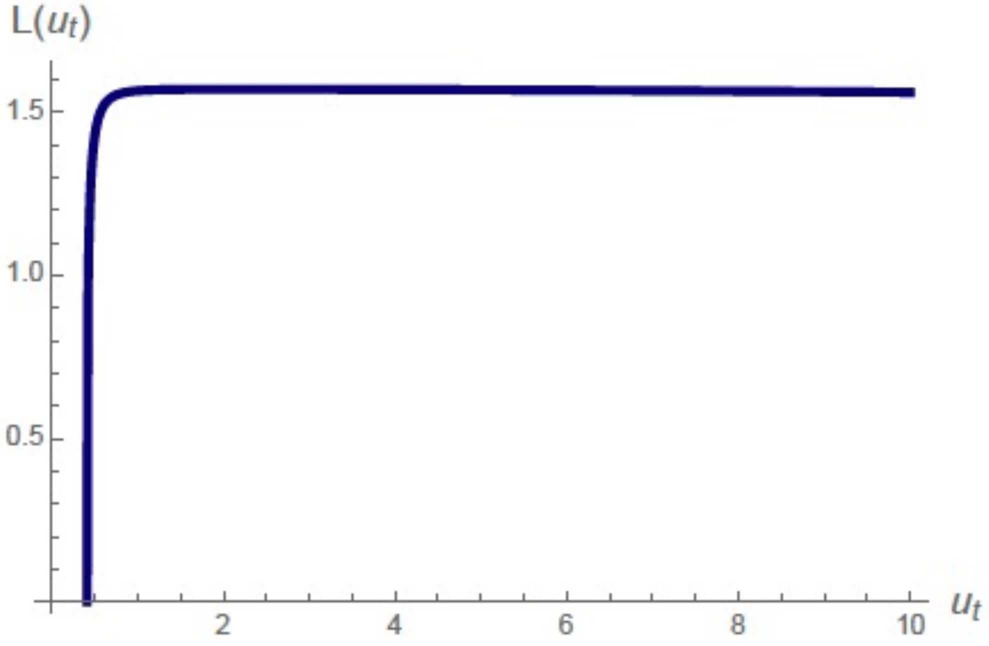
The domain wall structure consists probe D7 branes in a D5 brane geometry. The D5 geometry is

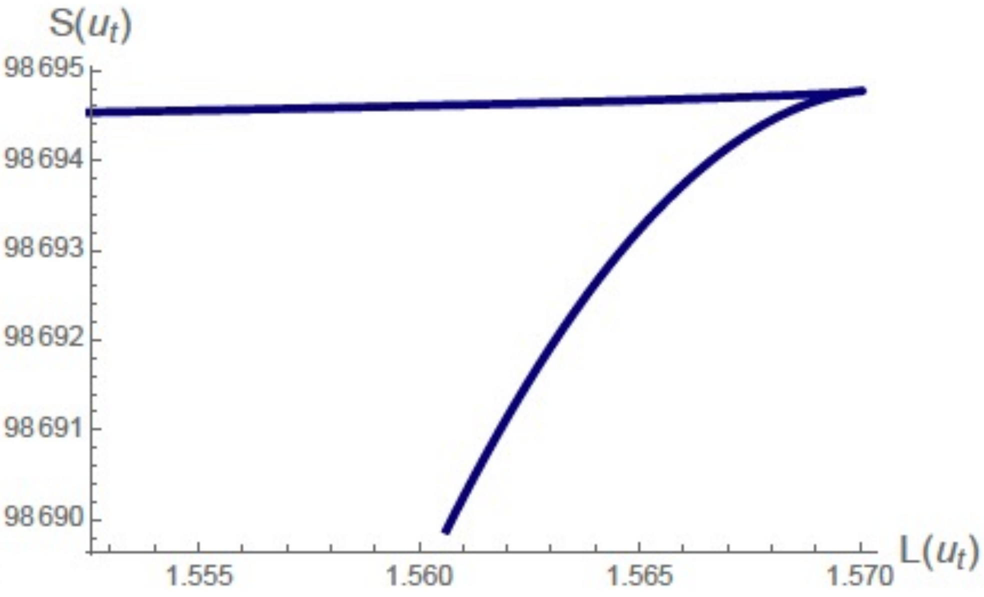
$$\boxed{ds^2 = \frac{u}{R}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dx_4^2 + dx_5^2 f(u;u_\Lambda)\right) + \frac{R}{u}\frac{du^2}{f(u;u_\Lambda)} + Rud\Omega_3^2,} \qquad f(u;u_\Lambda) = \left(1 - \frac{u_\Lambda^2}{u^2}\right)$$

$$e^\phi = g_s\frac{u}{R}, \qquad F_3 = \frac{2R^2}{g_s}\Omega_3 \qquad R^2 = g_s N_c \alpha'.$$

When one dimension of D5 geometry is being compactified, the confinement will come into play and affects the gauge degrees of freedom in the theory

Strips in D5 brane geometry



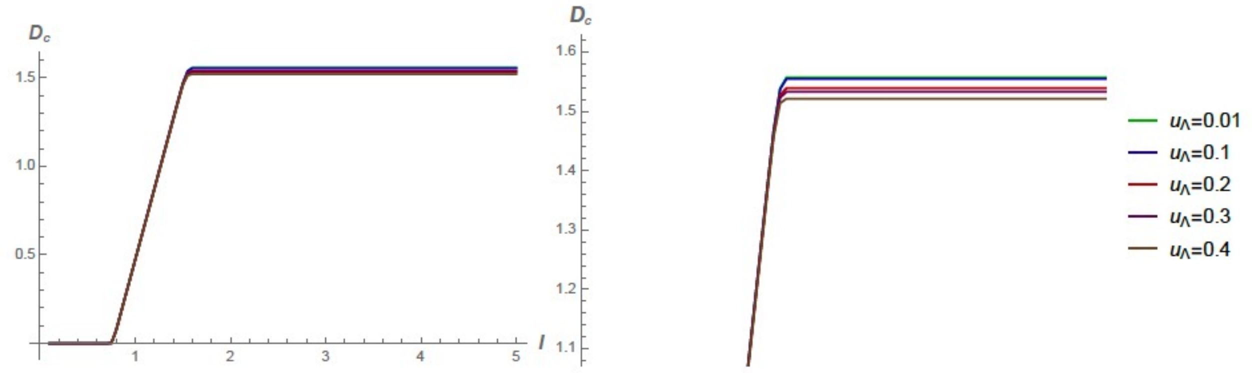


$$L(u_t) = 2R\int_{u_t}^{\infty}\frac{du}{u}\frac{1}{\sqrt{\left(1-\frac{u_\Lambda^2}{u^2}\right)\left(\frac{u^4}{u_t^4}\frac{1-\frac{u_\Lambda^2}{u^2}}{1-\frac{u_\Lambda^2}{u_t^2}}-1\right)}}.$$

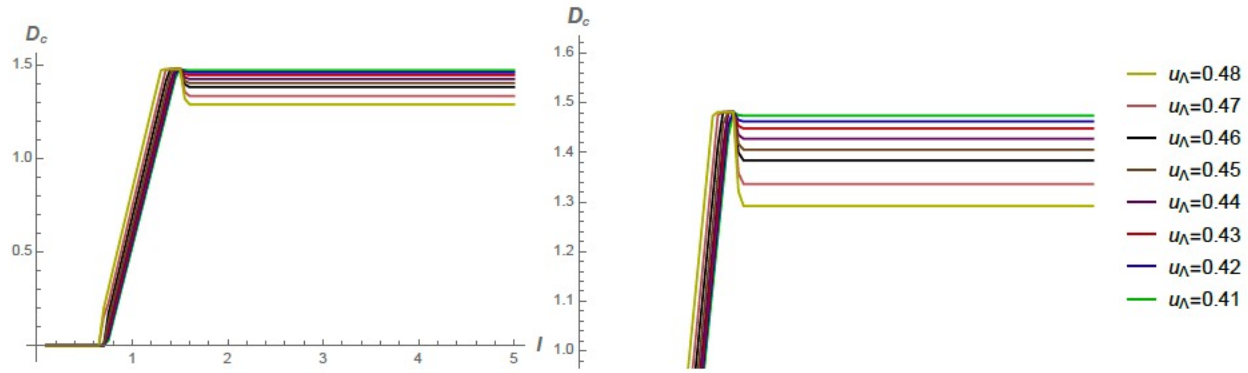
$$S_C(u_t) = \frac{(2\pi)^2 V_2 V_3}{2g_s^2 G_N^{(10)}}\int_{u_t}^{\infty} du \frac{u}{\sqrt{1-\frac{u_t^4}{u^4}\left(\frac{1-\frac{u_\Lambda^2}{u_t^2}}{1-\frac{u_\Lambda^2}{u^2}}\right)}},$$

D_c in D5 brane geometry

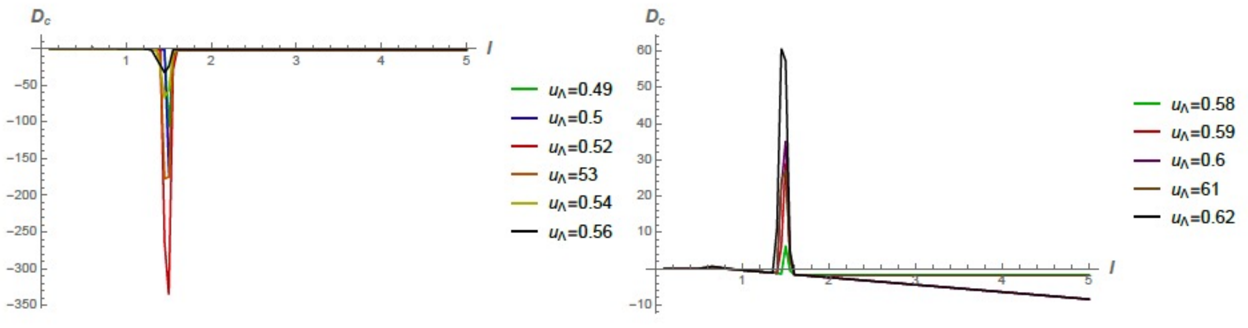
Phase 1



Phase 2



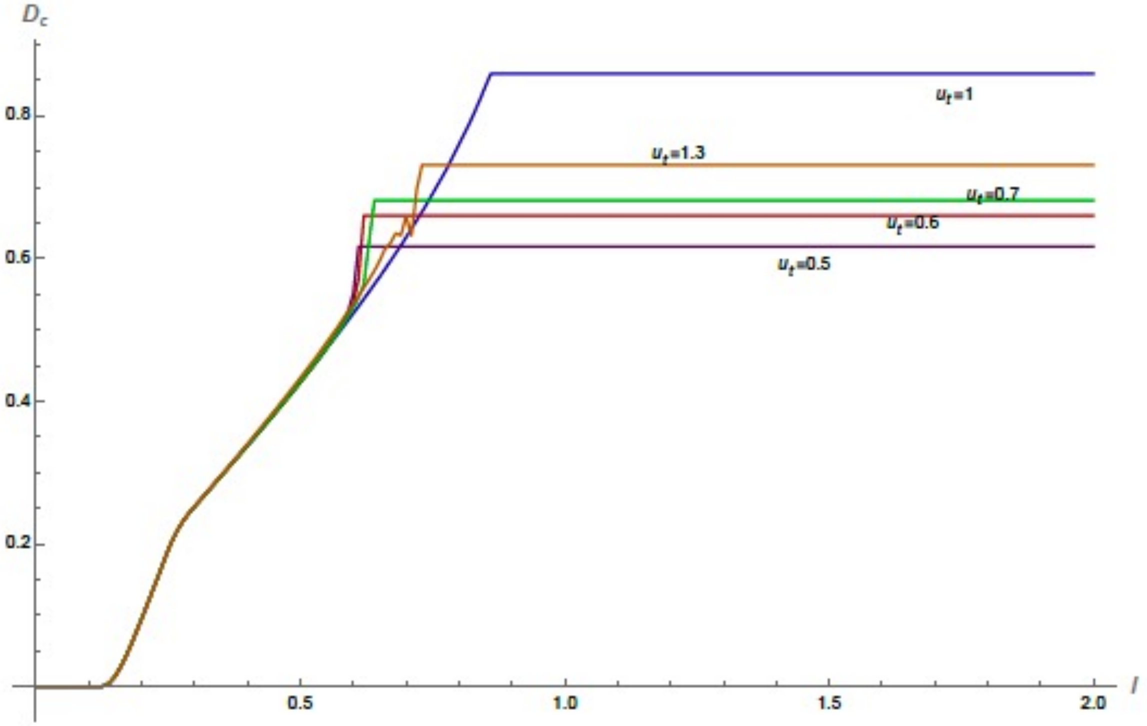
Phase 3



The other phase is more noisy and is not shown here.

$S_C - S_D$ Phase Diagram

One can also use $S_C - S_D$ instead of S_C which for the case of Witten-QCD for example would lead to the following phase diagram.



Again, phase transitions and phase transitions could be detetcted but no more physical information can be gained from this quantity, only the noises could become less.

- It would be interesting to compare the behaviour of quasi-normal modes (QNM) of D-branes and the geodesic motions in each of these models as in and check the connections between the quantum Sieberg Witten (SW) curves and gravitational perturbations dubbed SW-QNM correspondence.

- The behaviour of QNMs around the phase transitions and the connections with the critical distance, mutual information, EoP and negativity could be studied.

- The phase structures using other quantities, like various combinations of S, negativity, reflected entropy, etc could be constructed.

$$\mathcal{N} = \frac{3}{4}\left(2S(l+D) - S(2l+D) - S(D)\right)$$

$$\chi(\rho, \rho_i, p_i) \equiv S(\rho) - \sum_i p_i S(\rho_i),$$

$$\mathcal{N}(\rho) = \frac{||\rho^\Gamma||_1 - 1}{2},$$

$$S(A) = \max_{v^\mu} \int_A \sqrt{h} n_\mu v^\mu, \text{ s.t. } \nabla_\mu v^\mu = 0, |v^\mu| \leq 1,$$

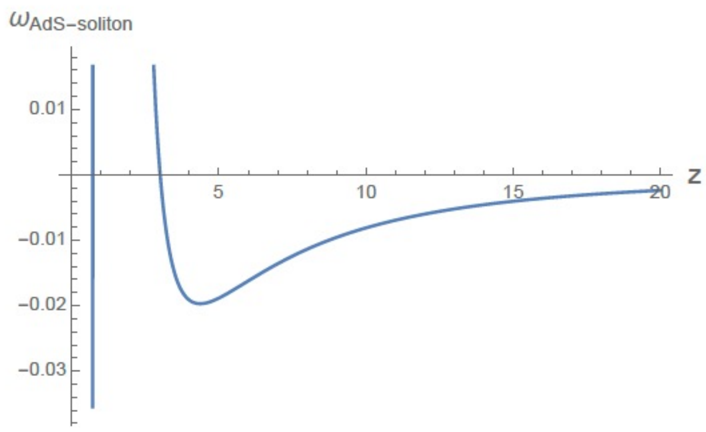
Crofton form in confining backgrounds

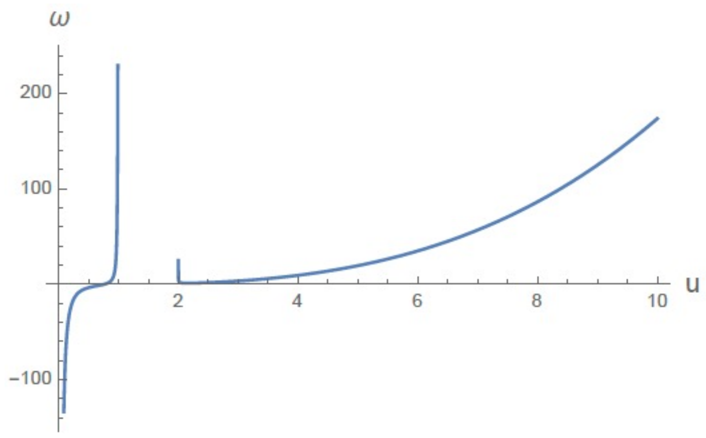
- Using ideas from integral geometry, one could find the connections between the length of a curve and the number of geodesics, or “random” lines it would be intersected and therefore its connection to the Crofton formula.

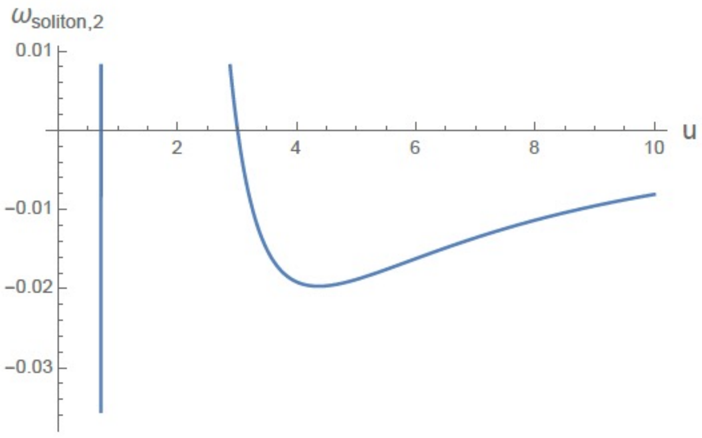
- This formula can give further intuitions about the structures of these various confining backgrounds from the perspective of bulk reconstruction and can shed more lights on their various specific properties such as singularities, throats, bulges, walls, etc and their effects on the correlations between the two strips in the mixed setup in such confining models.

Crofton form of AdS-Soliton

$$\omega_{\text{AdS-soliton}} = \frac{\left(2z_0 - \frac{(d-6)z^8\left(\frac{z}{z_0}\right)^{-d}}{z_0^7}\right)\left(1-\left(\frac{z_t}{z_0}\right)^{8-d}\right)}{2z_0 z_t^2\left(1-\left(\frac{z}{z_0}\right)^{8-d}\right)^2\left(1-\frac{z^2\left(1-\left(\frac{z_t}{z_0}\right)^{8-d}\right)}{\left(1-\left(\frac{z}{z_0}\right)^{8-d}\right)z_t^2}\right)^{\frac{3}{2}}} - \frac{1}{z^2\sqrt{1-\frac{z^2\left(1-\left(\frac{z_t}{z_0}\right)^{8-d}\right)}{\left(1-\left(\frac{z}{z_0}\right)^{8-d}\right)^2 z_t}}},$$

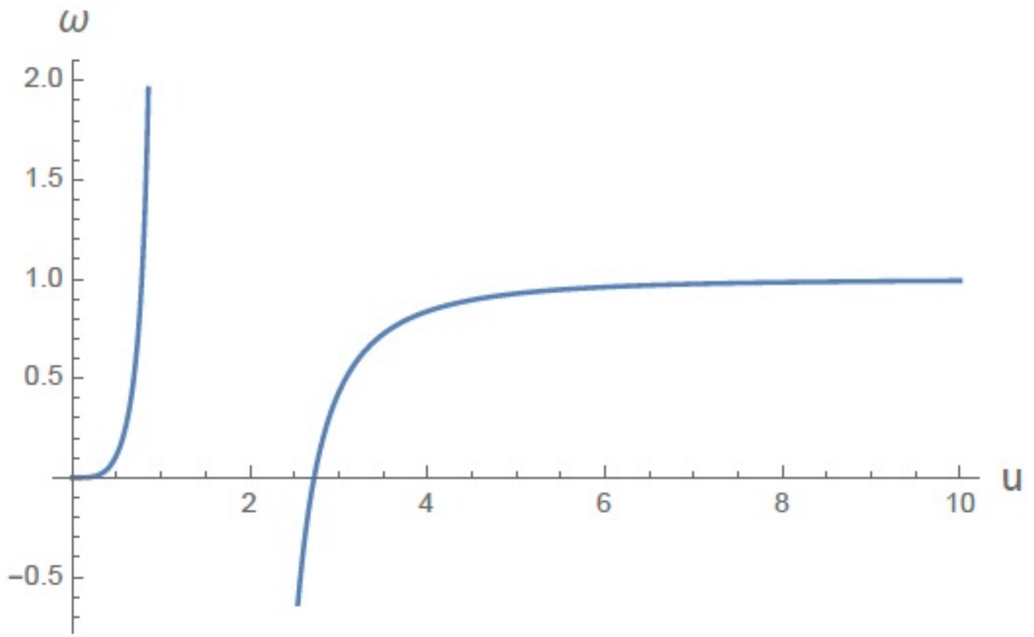






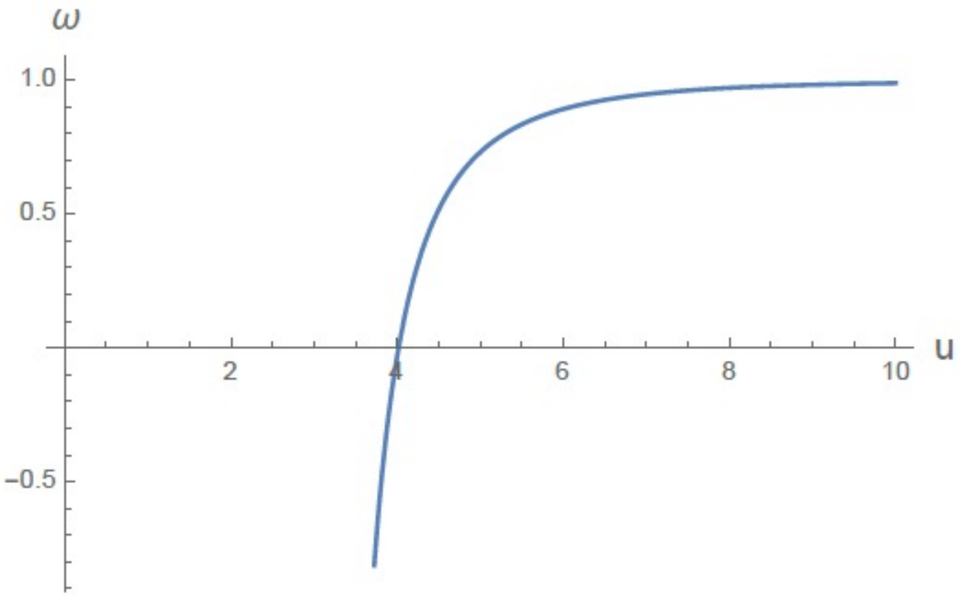
Crofton form of Sakai-Sugimoto

$$\omega_{\text{Sakai-Sugimoto}} = \frac{V_3 V_4 R_{D_4}^3}{2 g_s^2 G_N^{(10)}} \frac{u^4\left(10 u_{\text{KK}}^3 u_t^5 - 5u^5 u_{\text{KK}}^3 - 7u^3 u_t^5 + 2u^8\right)}{2\left((u^3 - u_{\text{KK}}^3)(u^5 - u_t^5)\right)^{\frac{3}{2}}} du \wedge d\theta.$$



Crofton form of deformed Sakai-Sugimoto

$$\omega_{\text{deformed Sakai-Sugimoto}} = \frac{V_3 V_4 R_{D_4}^3}{2 g_s^2 G_N^{(10)}} \frac{u^4}{2\left(\frac{u^8 - u^5 u_{\text{KK}}^3}{u_0^8 - u_0^5 u_{\text{KK}}^3} - 1\right)^{\frac{1}{2}} \left(u^3 - u_{\text{KK}}^3\right)^{\frac{3}{2}} \left(u^5 - u_t\right)^{\frac{3}{2}}} \times$$
$$\left(\frac{R_{D_4}^{\frac{3}{2}} (3u_0^8 u^3 - (2u^5 + 3u_0^5) u_{\text{KK}}^6 + (13u^8 - 3u_0^5 u^3 + 3u_0^8) u_{\text{KK}}^3 - 11u^{11})(u^5 - u_t^5)}{u_0^5 u_{\text{KK}}^3 - u^5 u_{\text{KK}}^3 + u^8 - u_0^8} \right.$$
$$+ R_{D_4}^{\frac{3}{2}} \left(7u_{\text{KK}}^3 u_t^5 - 2u^5 u_{\text{KK}}^3 - 10u^3 u_t^5 + 5u^8\right)$$
$$\left. + \left(\frac{u^8 - u^5 u_{\text{KK}}^3}{u_0^8 - u_0^5 u_{\text{KK}}^3} - 1\right)^{\frac{1}{2}} \left(10u_{\text{KK}}^3 u_t^5 - 5u^5 u_{\text{KK}}^3 - 7u^3 u_t^5 + 2u^8\right) \right) du \wedge d\theta,$$



One could see that for intermediate and bigger u the behaviour is similar to the Sakai- Sugimoto case, as one would expect.

Crofton form of Klebanov-Strassler

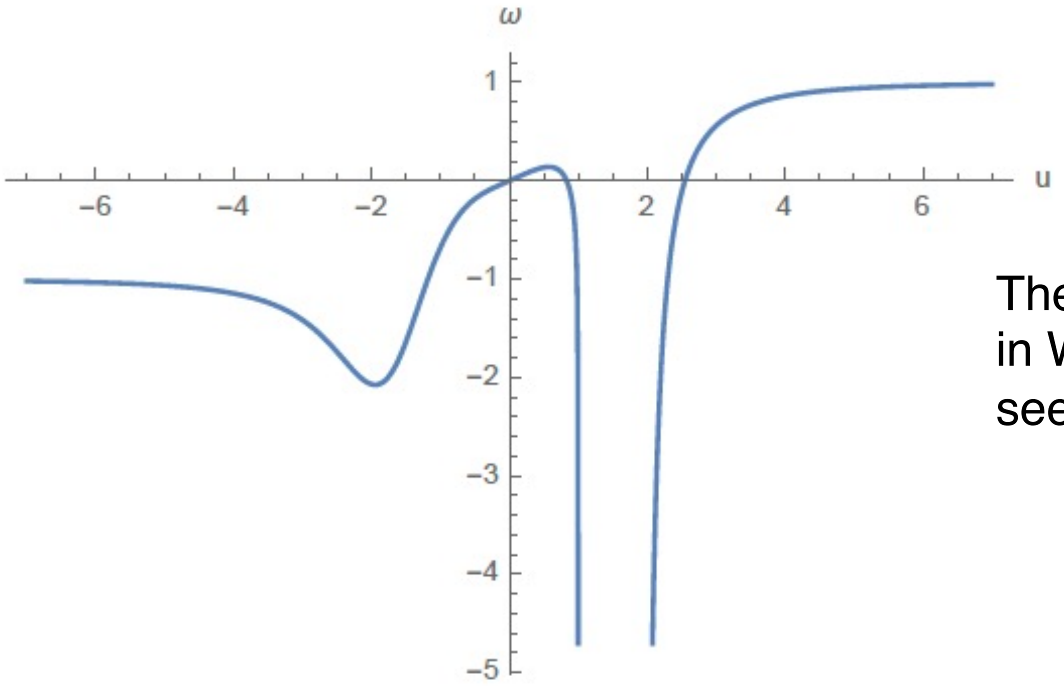
$$\omega_{\mathrm{KS}} = \frac{2^{2/3}\pi^3 V_2 \epsilon^{-8/3}\epsilon^4(\alpha_p g_s M)^2}{3G_N} \frac{h(\tau)\sinh(2\tau)}{\sqrt{1-\sinh^2(\tau_0)\,\mathrm{csch}^2(\tau)\left(\frac{\tau_0-\sinh(\tau_0)\cosh(\tau_0)}{\tau-\sinh(\tau)\cosh(\tau)}\right)^{\frac{2}{3}}}} -$$

$$\frac{h(\tau)\sinh^2(\tau_0)\left(2\tau_0-\sinh(2\tau_0)\right)\left(12\tau\coth(\tau)-5\sinh(3\tau)\mathrm{csch}(\tau)+3\right)}{6(\sinh(2\tau)-2\tau)^2\sqrt[3]{\frac{\tau_0-\sinh(\tau_0)\cosh(\tau_0)}{\tau-\sinh(\tau)\cosh(\tau)}}\left(1-\sinh^2(\tau_0)\,\mathrm{csch}^2(\tau)\left(\frac{\tau_0-\sinh(\tau_0)\cosh(\tau_0)}{\tau-\sinh(\tau)\cosh(\tau)}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}}} +$$

$$\frac{\sinh^2(\tau)h'(\tau)}{\sqrt{1-\sinh^2(\tau_0)\,\mathrm{csch}^2(\tau)\left(\frac{\tau_0-\sinh(\tau_0)\cosh(\tau_0)}{\tau-\sinh(\tau)\cosh(\tau)}\right)^{\frac{2}{3}}}}.$$

Crofton form of Witten-QCD

$$\omega = \frac{-1}{2}\partial_u^2 S du \wedge d\theta =$$
$$\frac{V_2}{G_N^{(10)}}\frac{4\pi^2 R^{\frac{9}{2}}}{9g_s^2\sqrt{u_t}}\left(\frac{u\left(2\left(u^2-2u_0^2\right)u_t^6+\left(7u_0^2u^3+4u_0^5-4u^5\right)u_t^3+2u^8-7u_0^5u^3\right)}{2\left(u^5-u_0^5-u^2\left(u_t^3-u_0^2\right)\right)^{\frac{3}{2}}\sqrt{u^3-u_t^3}}\right)du\wedge d\theta$$

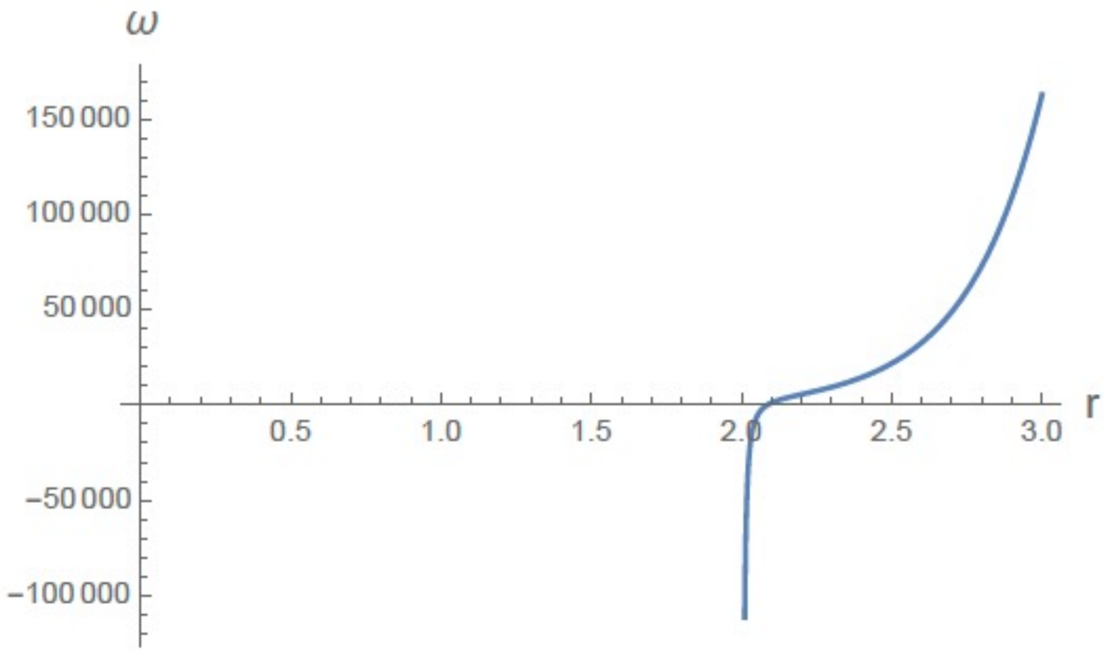


The behaviour of Crofton form for $u_0 = 2$ and $u_t = 1$ in WQCD background is shown where one can see that around u_t and u_0 we have an infinite well.

Crofton form of Maldacena-Nunez

$$\boxed{\omega_{MN} = c_{MN} \frac{2\sinh(4r)\left(\sinh^4(2r) - 2\sinh^4(2r_0)\right)}{\left(\sinh^4(2r) - \sinh^4(2r_0)\right)\sqrt{1 - \sinh^4(2r_0)\,\mathrm{csch}^4(2r)}},}$$

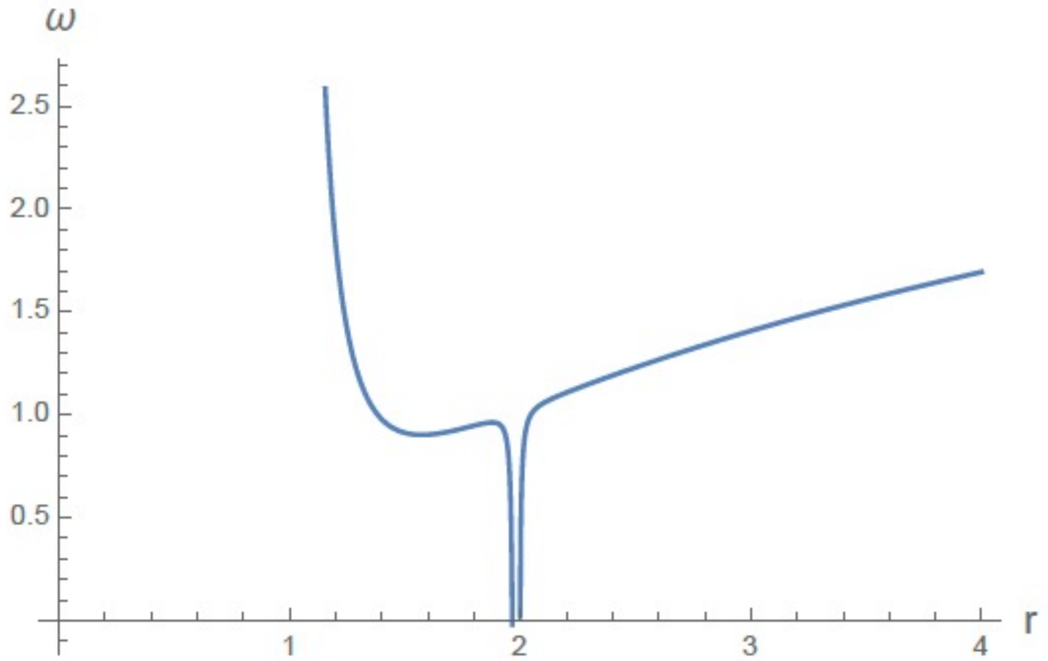
$$c_{MN} = \frac{V_2 \pi^3 e^{4\phi_0}}{G_N^{(10)}}.$$



Crofton form of Klebanov-Tseytlin

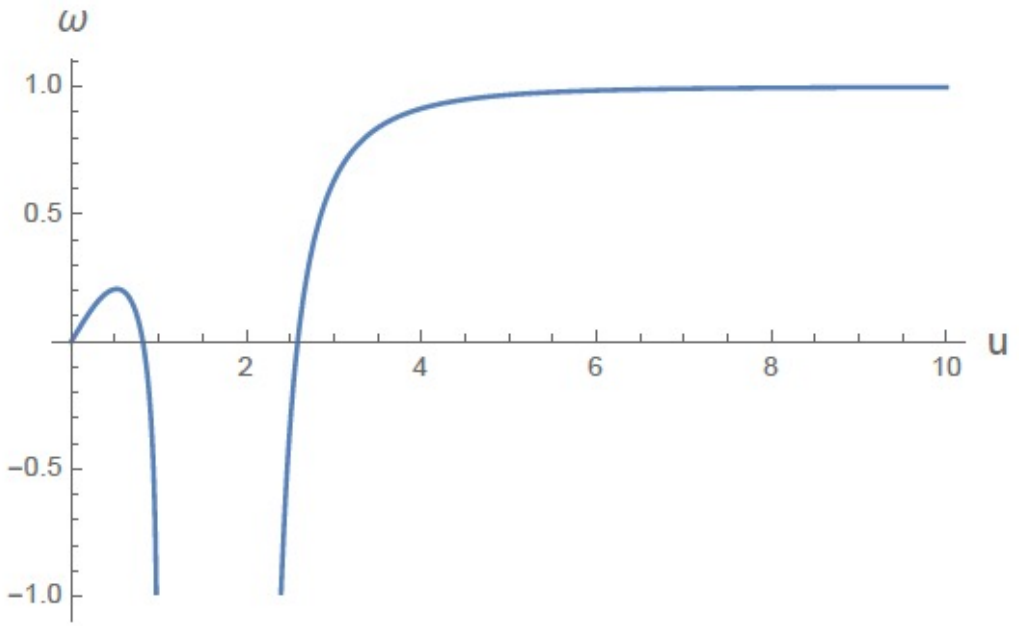
$$\boxed{\omega_{KT} = c_{KT}\frac{2r^9\log^{\frac{3}{2}}(\frac{r}{r_s})\left(\log(\frac{r}{r_s})+1\right) - r_0^6 r^3\log(\frac{r_0}{r_s})\log^{\frac{1}{2}}(\frac{r}{r_s})\left(8\log(\frac{r}{r_s})+3\right)}{2\left(r^6\log\left(\frac{r}{r_s}\right) - r_0^6\log\left(\frac{r_0}{r_s}\right)\right)^{\frac{3}{2}}},}$$

$$c_{KT} = \frac{12V_2\pi^3M^2g_s\epsilon^4}{G_N^{(10)}}$$



Crofton form of Domain-Wall QCD (D5 brane)

$$\boxed{\omega = \frac{u\left(u^2\left(3u_t^2u_\Lambda^2 - 3u_t^4 + u_\Lambda^4\right) + 2u_t^2u_\Lambda^2\left(u_t^2 - u_\Lambda^2\right) + u^6 - 2u^4u_\Lambda^2\right)}{\left(u^2 - u_\Lambda^2\right)^{\frac{1}{2}}\left(\left(u^2 - u_t^2\right)\left(u_t^2 + u^2 - u_\Lambda^2\right)\right)^{\frac{3}{2}}}}$$



Again, the hole around the u_Λ and u_t could be detected while for larger u it would become constant and again it would point to the structure of this background as the Crofron form could be used as a holographic tool in the bulk reconstruction.

Thank You!

Modular Berry Connection for Celestial Field theories

Future work!

Modular connection for CFT vacuum

arXiv:1903.04493

The two-sided modular Hamiltonian for an interval in the CFT vacuum can be written in terms of the conformal generators as

$$H_{\mathrm{mod}} = K_+ + K_-$$

$$K_+ = s_1 L_1 + s_0 L_0 + s_{-1} L_{-1}$$
$$K_- = t_1 \bar{L}_1 + t_0 \bar{L}_0 + t_{-1} \bar{L}_{-1}.$$

The coefficients s_i, t_i are determined, up to an overall multiplicative constant, by the requirement that the generators K_+ and K_- preserve the left-moving and right-moving null coordinates of the interval endpoints (a^+, b^+) and (a^- , b^-), respectively.

CFT global generators are $\boxed{L_{-1} = ie^{-ix^+}\partial_+ \quad \text{and} \quad L_0 = i\partial_+ \quad \text{and} \quad L_1 = ie^{ix^+}\partial_+,}$

with an identical action of the L_i is on the x^- null coordinate, one finds:

$$
\begin{aligned}
s_1 &= \frac{2\pi\cot(b^+ - a^+)/2}{e^{ia^+} + e^{ib^+}} \\
s_0 &= -2\pi\cot(b^+ - a^+)/2 \\
s_{-1} &= \frac{2\pi\cot(b^+ - a^+)/2}{e^{-ia^+} + e^{-ib^+}}
\end{aligned}
\qquad
\begin{aligned}
t_1 &= -\frac{2\pi\cot(b^- - a^-)/2}{e^{ia^-} + e^{ib^-}} \\
t_0 &= 2\pi\cot(b^- - a^-)/2 \\
t_{-1} &= -\frac{2\pi\cot(b^- - a^-)/2}{e^{-ia^-} + e^{-ib^-}}
\end{aligned}
$$

Then following the procedure in 1903.04493, from Modular Hamiltonian, one can read Modular Berry connection and Modular Berry curvature and the bound on Modular Chaos.

Modular Hamiltonian in BMSFTs

arXiv:2006.10741

Vacuum structure in Poincare coordinate

$$ds^2 = -2dudr + r^2 dz^2,$$

Interval

$$\partial\mathcal{A} = \{(u_-, z_-), (u_+, z_+)\}, \qquad l_u \equiv u_+ - u_-, \quad l_z \equiv z_+ - z_-,$$

Modular flow generator as the linear combination of vacuum symmetry generators:

$$\zeta = \zeta^u \partial_u + \zeta^z \partial_z = \sum_{j=-1}^{1} (a_j \ell_j + b_j m_j),$$

Vacuum symmetry generators:

$$\boxed{\ell_j = -z^{j+1}\partial_z - (j+1) z^j u \partial_u, \qquad m_j = -z^{j+1}\partial_u,}$$

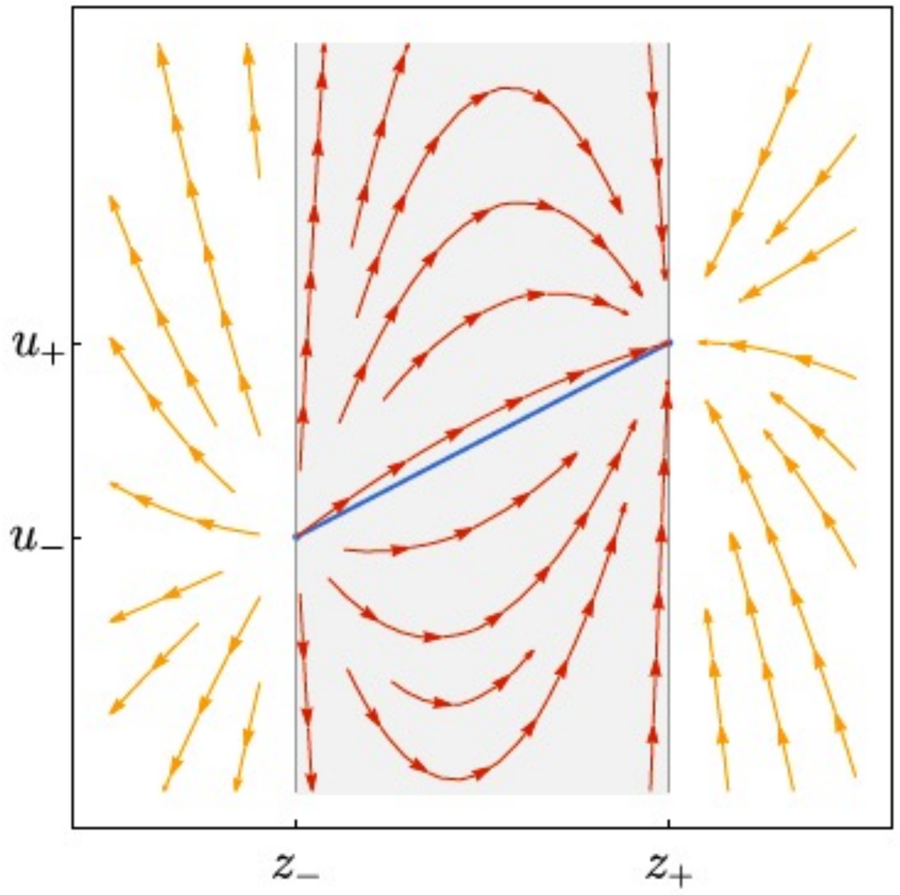
$$(a_+, a_0, a_-) = a_+\big(1, -z_- - z_+, z_+z_-\big),$$
$$(b_+, b_0, b_-) = b_+\big(1, -z_- - z_+, z_+z_-\big) + a_+\big(0, -u_- - u_+, u_+z_- + u_-z_+\big).$$

$$\partial_s u(s) = \zeta^u, \qquad \partial_s z(s) = \zeta^z,$$

$$\rightarrow \quad \begin{aligned}(a_+, a_0, a_-) &= \frac{2\pi}{z_+ - z_-}(1, -z_- - z_+, z_+z_-),\\ (b_+, b_0, b_-) &= \frac{2\pi}{(z_+ - z_-)^2}(u_- - u_+, 2u_+z_- - 2u_-z_+, u_-z_+^2 - u_+z_-^2).\end{aligned}$$

$$\boxed{\zeta = \sum_{j=-1}^{1}(a_j\ell_j + b_jm_j) = \big[T(z) + uY'(z)\big]\partial_u + Y(z)\partial_z,}$$

Modular Flow in BMSFTs



arXiv:2006.10741

Modular Hamiltonian in WCFTs

$$\partial\mathcal{A} = \{(z_-, w_-), (z_+, w_+)\}, \qquad l_z \equiv z_+ - z_-, \quad l_w \equiv w_+ - w_-,$$

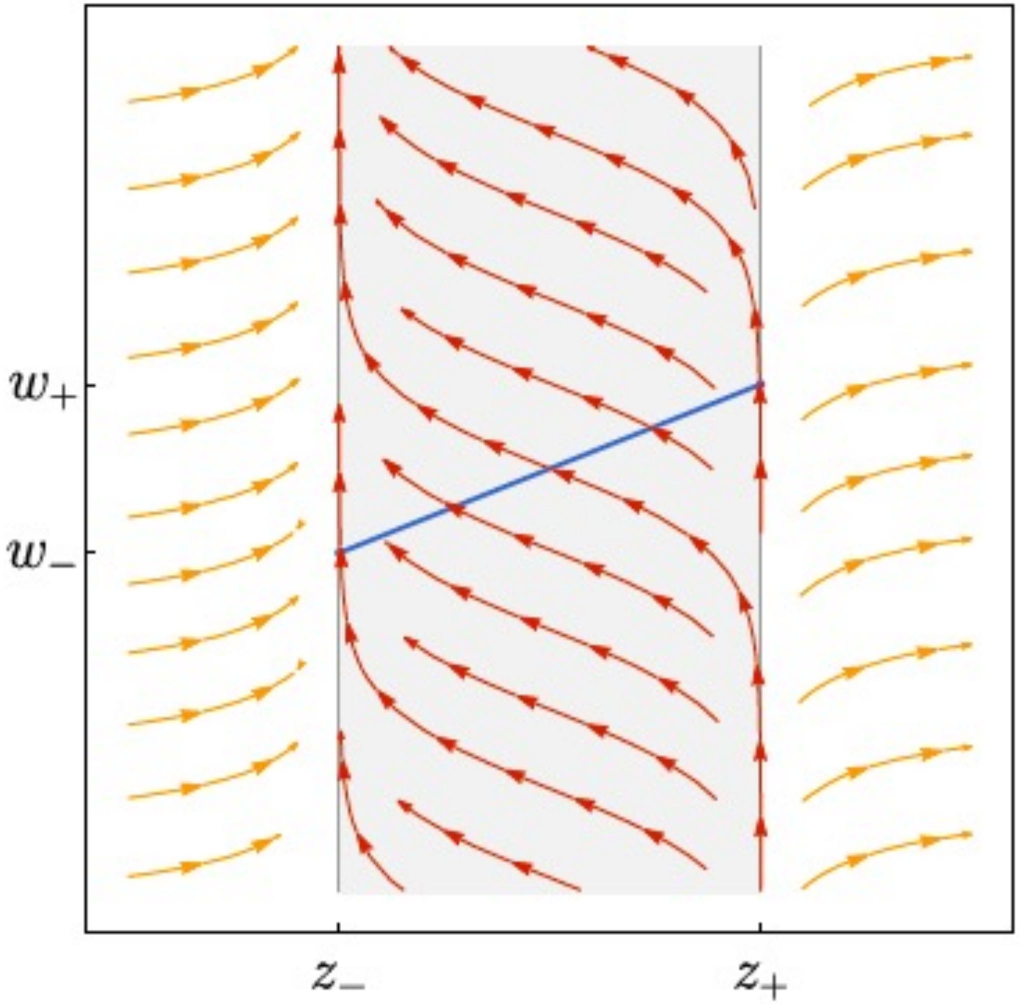
global $SL(2,R) \times U(1)$ generators $\boxed{\ell_1 = -z^2\partial_z, \qquad \ell_0 = -z\partial_z, \qquad \ell_{-1} = -\partial_z, \qquad \bar{\ell}_0 = -\partial_w,}$

$$\zeta = \sum_{i=-1}^{1} a_i \ell_i + \bar{a}_0 \bar{\ell}_0.$$

$$a_1 = 2\pi\frac{1}{z_+ - z_-}, \qquad a_0 = -2\pi\frac{z_+ + z_-}{z_+ - z_-}, \qquad a_{-1} = 2\pi\frac{z_+ z_-}{z_+ - z_-}.$$

$$\zeta = 2\pi\mu\bar{\ell}_0 + \frac{2\pi}{z_+ - z_-}[\ell_1 - (z_+ + z_-)\ell_0 + z_+ z_- \ell_{-1} = -2\pi\mu\partial_w - \frac{2\pi}{z_+ - z_-}[z_+ z_- - (z_+ + z_-)z + z^2]\partial_z.$$

Modular Flow in WCFs



arXiv:2006.10741

Modular Hamiltonian in Celestial Filed theories?

Symmetries:

The "Lorentz group" of $\mathbb{K}^{2,2}$ is $SO(2,2) \cong \frac{SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R}{\mathbb{Z}_2}$, where the $\mathbb{Z}_2$ is generated by $-1_L \times -1_R$. The spin group is the double cover $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$. The symmetry is generated on Klein space by (real combinations of) the six Killing vector fields

$$L_1 = \bar{z}\partial_w + \bar{w}\partial_z, \qquad \bar{L}_1 = z\partial_w + \bar{w}\partial_{\bar{z}},$$

$$L_0 = \frac{1}{2}\left(z\partial_z + w\partial_w - \bar{z}\partial_{\bar{z}} - \bar{w}\partial_{\bar{w}}\right), \quad \bar{L}_0 = \frac{1}{2}\left(-z\partial_z + w\partial_w + \bar{z}\partial_{\bar{z}} - \bar{w}\partial_{\bar{w}}\right),$$

$$L_{-1} = -z\partial_{\bar{w}} - w\partial_{\bar{z}}, \qquad \bar{L}_{-1} = -\bar{z}\partial_{\bar{w}} - w\partial_z.$$

arXiv:2101.09591

$$L_1 = \frac{1}{2}e^{-i\psi-i\phi}\left(\partial_\rho - i\tanh\rho\,\partial_\psi - i\coth\rho\,\partial_\phi\right),$$
$$L_0 = -\frac{i}{2}\left(\partial_\psi + \partial_\phi\right),$$
$$L_{-1} = \frac{1}{2}e^{i\psi+i\phi}\left(-\partial_\rho - i\tanh\rho\,\partial_\psi - i\coth\rho\,\partial_\phi\right),$$
$$\bar{L}_1 = \frac{1}{2}e^{-i\psi+i\phi}\left(\partial_\rho - i\tanh\rho\,\partial_\psi + i\coth\rho\,\partial_\phi\right),$$
$$\bar{L}_0 = -\frac{i}{2}\left(\partial_\psi - \partial_\phi\right),$$
$$\bar{L}_{-1} = \frac{1}{2}e^{i\psi-i\phi}\left(-\partial_\rho - i\tanh\rho\,\partial_\psi + i\coth\rho\,\partial_\phi\right),$$

$$L_1 = \frac{1}{2}e^{-i\psi-i\phi}\left(\partial_{\tilde\rho} - i\coth\tilde\rho\,\partial_\psi - i\tanh\tilde\rho\,\partial_\phi\right),$$
$$L_0 = -\frac{i}{2}\left(\partial_\psi + \partial_\phi\right),$$
$$L_{-1} = \frac{1}{2}e^{i\psi+i\phi}\left(-\partial_{\tilde\rho} - i\coth\tilde\rho\,\partial_\psi - i\tanh\tilde\rho\,\partial_\phi\right),$$
$$\bar{L}_1 = \frac{1}{2}e^{-i\psi+i\phi}\left(\partial_{\tilde\rho} - i\coth\tilde\rho\,\partial_\psi + i\tanh\tilde\rho\,\partial_\phi\right),$$
$$\bar{L}_0 = -\frac{i}{2}\left(\partial_\psi - \partial_\phi\right),$$
$$\bar{L}_{-1} = \frac{1}{2}e^{i\psi-i\phi}\left(-\partial_{\tilde\rho} - i\coth\tilde\rho\,\partial_\psi + i\tanh\tilde\rho\,\partial_\phi\right).$$

Now, using these global generators, how to find Modular Hamiltonian and Modular Flow for Celestial Field theories??!